Invariants of quivers under the action of classical groups.

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Abstract

We consider a generalization of representations of quivers that can be derived from the ordinary representations of quivers by considering a product of arbitrary classical groups instead of a product of the general linear groups and by considering the dual action of groups on "vertex" vector spaces together with the usual action. A generating system for the corresponding algebra of invariants is found. In particular, a generating system for the algebra of $SO(n)$-invariants of several matrices is constructed over a field of characteristic different from 2. The proof uses the reduction to semi-invariants of mixed representations of a quiver and the decomposition formula that generalizes Amitsur's formula for the determinant.

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1 Introduction

We work over an infinite field K of arbitrary characteristic. All vector spaces, algebras, and modules are over K unless otherwise stated.

A *quiver* is a finite oriented graph. This notion was introduced by Gabriel in [17] as an effective mean for description of different problems of the linear algebra. The importance of this notion from the point of view of the representational theory is due to the fact that the category of representations of a quiver is equivalent to the category of finite dimensional modules over the path algebra associated with the quiver. Since every finite dimensional basic algebra over algebraically closed field is a factor-algebra of the path algebra of some quiver (see Chapter 3 from [14]), the category of finite dimensional modules over such an algebra is a full subcategory of the category of representations of the quiver. Invariants of quivers are important not only in the invariant theory but also in the representational theory because these invariants distinguish semi-simple representations of a quiver.

A representation of a quiver with l vertices consists of a collection of column vector spaces $K^{\boldsymbol{n}_1},\ldots,K^{\boldsymbol{n}_l}$, assigned to the vertices, and linear mappings between the vector spaces "along" the arrows. We generalize this notion as follows. Let v, where $1\leq v\leq l$, be a vertex of the quiver. In the classical case $GL(\boldsymbol{n}_v)$ acts on $K^{\boldsymbol{n}_v}$ but in our case an arbitrary classical group from the list $GL(\boldsymbol{n}_v)$, $O(\boldsymbol{n}_v)$, $Sp(\boldsymbol{n}_v)$, $SL(\boldsymbol{n}_v)$, $SO(\boldsymbol{n}_v)$ can act on $K^{\boldsymbol{n}_v}$. Moreover, we consider the dual space $(K^{\boldsymbol{n}_v})^*$ together with $K^{\boldsymbol{n}_v}$ in order to deal with bilinear forms together with linear mappings. Finally, instead of arbitrary linear mappings "along" arrows we consider only those that, for example, preserve some bilinear symmetric form on "vertex" spaces, etc. This construction is called a *mixed quiver setting*. The exact definition together with examples is given in Section 2.1.

Orthogonal and *symplectic* representations of *symmetric* quivers, *(super)mixed* representations of quivers, and representations of *signed* quivers, respectively, introduced by Derksen and Weyman in [5], Zubkov in [35], and Shmelkin in [28], respectively, are partial cases of this construction (see part 3 of Example 1). The motivation for these generalizations of quivers from the point of view of the representational theory of algebraic groups was given in [5], [28], where symmetric and signed quivers, respectively, of tame and finite type were classified.

In the paper we established generators for the invariants of a mixed quiver setting (see Theorem 1). In particular, in Section 4.2 we completed description, originated by Sibirskii in [29] and Procesi in [26], of generators for the invariants of several matrices under the diagonal action by conjugation of a classical group.

The paper is organized as follows.

Section 2 is started with the definition of generalized quivers and their representations. It is followed by an overview of known results on generating systems for the invariants of quivers.

Section 3 contains notations that are used throughout the paper. We also recall some definitions from [24] such as a block partial linearization of the pfaffian (b.p.l.p.) and a tableau with substitution.

In Section 4 our main result (Theorem 1) is formulated in terms of b.p.l.p.-s, where non-zero blocks are "generic" matrices. At the end of the section we consider the quiver with one vertex (Corollary 2). This special case is of great importance since it is the simplest one and, roughly speaking, the general case can be reduced to it.

Sections 5–9 are dedicated to the proof of Theorem 1. In Section 5 we show that elements from Theorem 1 are invariants (Lemma 4). In Section 6 we rewrite generators of semi-invariants from [23] in terms of b.p.l.p.-s, and thus obtain a set generating the algebra of GL- and SL-invariants as a vector space over K (Theorem 2). The general case is reduced to GL- and SL-invariants in Section 7 by means of Frobenius reciprocity and the theory of modules with good filtration. In Theorem 7 we show that the space of invariants is the image of GL- and SL-invariants of an explicitly constructed quiver. In Section 8 we apply the decomposition formula from [24] to rewrite a b.p.l.p. as a polynomial in b.p.l.p.-s of a special form (Theorem 9). Using this, in Section 9 we describe the image of GL- and SL-invariants and show that elements from Theorem 1 generate the algebra of invariants.

2 Generalized representations of quivers

2.1 Definitions

A *quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a finite oriented graph, where $\mathcal{Q}_0 = \{1, \ldots, l\}$ is the set of vertices and $\mathcal{Q}_1$ is the set of arrows. For an arrow α, denote by α' its head and by α'' its tail. Given a *dimension vector* $\boldsymbol{n} = (\boldsymbol{n}_1, \ldots, \boldsymbol{n}_l)$, we assign an $\boldsymbol{n}_v$-dimensional vector space V_v to $v \in \mathcal{Q}_0$. We identify V_v with the space of column vectors $K^{\boldsymbol{n}_v}$. Fix the *standard* basis $e(v,1), \ldots, e(v, \boldsymbol{n}_v)$ for $K^{\boldsymbol{n}_v}$, where $e(v,i)$ is a column vector whose i-th entry is 1 and the rest of entries are zero. A *representation* of $\mathcal{Q}$ of dimension vector $\boldsymbol{n}$ is a collection of matrices

$$h = (h_\alpha)_{\alpha \in \mathcal{Q}_1} \in H = H(\mathcal{Q}, \boldsymbol{n}) = \bigoplus_{\alpha \in \mathcal{Q}_1} K^{\boldsymbol{n}_{\alpha'} \times \boldsymbol{n}_{\alpha''}} \simeq \bigoplus_{\alpha \in \mathcal{Q}_1} \mathrm{Hom}_K(V_{\alpha''}, V_{\alpha'}),$$

where $K^{n_1\times n_2}$ stands for the linear space of $n_1\times n_2$ matrices over K and the isomorphism is given by the choice of bases. We will refer to H as the *space of representations* of $\mathcal{Q}$ of dimension vector $\boldsymbol{n}$. The action of the group

$$G=GL(\boldsymbol{n})=\prod_{v\in\mathcal{Q}_0}GL(\boldsymbol{n}_v)$$

on H is via the change of the bases for V_v ($v\in\mathcal{Q}_0$). In other words, $GL(\boldsymbol{n}_v)$ acts on V_v by left multiplication, and this action induces the action of G on H by

$$g\cdot h=(g_{\alpha'}h_\alpha g_{\alpha''}^{-1})_{\alpha\in\mathcal{Q}_1},$$

where $g=(g_\alpha)_{\alpha\in\mathcal{Q}_1}\in G$ and $h=(h_\alpha)_{\alpha\in\mathcal{Q}_1}\in H$.

The coordinate ring of the affine variety H is the polynomial ring

$$K[H]=K[x_{ij}^\alpha\,|\,\alpha\in\mathcal{Q}_1,\,1\le i\le\boldsymbol{n}_{\alpha'},1\le j\le\boldsymbol{n}_{\alpha''}].$$

Here x_{ij}^α stands for the coordinate function on H that takes a representation $h\in H$ to the (i,j)-th entry of a matrix h_α from $K^{\boldsymbol{n}_{\alpha'}\times\boldsymbol{n}_{\alpha''}}$. Denote by $X_\alpha=(x_{ij}^\alpha)$ the $\boldsymbol{n}_{\alpha'}\times\boldsymbol{n}_{\alpha''}$ *generic* matrix.

We will use the following notation to define the action of G on $K[H]$. Given $g\in G$, we write $g\cdot X_\alpha$ for the matrix, whose (i,j)-th entry is $g\cdot x_{ij}^\alpha$. Similarly, $\Phi(X_\alpha)$ stands for the matrix, whose (i,j)-th entry is $\Phi(x_{ij}^\alpha)$, where Φ is a mapping, defined on $K[H]$.

The action of G on H induces the action on $K[H]$ as follows: $(g\cdot f)(h)=f(g^{-1}\cdot h)$ for all $g\in G$, $f\in K[H]$, $h\in H$. In other words,

$$g\cdot X_\alpha=g_{\alpha'}^{-1}X_\alpha g_{\alpha''}$$

for $g\in G$, $\alpha\in\mathcal{Q}_1$. The algebra of *invariants* is

$$K[H]^G=\{f\in K[H]\,|\,g\cdot f=f\text{ for all }g\in G\}.$$

Given a positive integer n, let us fix the following notations for the classical groups:

$O(n)=\{A\in K^{n\times n}\,|\,AA^t=A^tA=E\}$, $Sp(2n)=\{A\in K^{2n\times 2n}\,|\,A^tJA=J\}$, $SO(n)=\{A\in O(n)\,|\,\det(A)=1\}$, where $E=E(n)$ is the identity matrix, $J=J(2n)=\left(\begin{array}{cc}0&E(n)\\-E(n)&0\end{array}\right)$ is the matrix of the skew-symmetric bilinear form on K^{2n};

and for certain subspaces of $K^{n\times n}$:

> $S^+(n) = \{A \in K^{n\times n} \mid A^t = A\}$ is the space of symmetric matrices, $S^-(n) = \{A \in K^{n\times n} \mid A^t = -A\}$ is the space of skew-symmetric matrices, $L^+(n) = \{A \in K^{n\times n} \mid AJ$ is a symmetric matrix$\}$, $L^-(n) = \{A \in K^{n\times n} \mid AJ$ is a skew-symmetric matrix$\}$.

The notion of representations of quivers can be generalized by the successive realization of the following steps.

1. Instead of $GL(\boldsymbol{n})$ we can take a product $G(\boldsymbol{n},\boldsymbol{g})$ of classical linear groups. Here $\boldsymbol{g} = (\boldsymbol{g}_1,\ldots,\boldsymbol{g}_l)$ is a vector, whose entries $\boldsymbol{g}_1,\ldots,\boldsymbol{g}_l$ are symbols from the list GL, O, Sp, SL, SO. By definition,

$$G(\boldsymbol{n},\boldsymbol{g}) = \prod_{v\in\mathcal{Q}_0} G_v,$$

   where

$$G_v = \begin{cases} GL(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = GL \\ O(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = O \\ Sp(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = Sp \\ SL(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = SL \\ SO(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = SO \end{cases}.$$

   Obviously, we have to assume that $\boldsymbol{n}$ and $\boldsymbol{g}$ are subject to the following restrictions:

   a) if $\boldsymbol{g}_v = Sp$ ($v \in \mathcal{Q}_0$), then $\boldsymbol{n}_v$ is even;

   b) if $\boldsymbol{g}_v$ is O or SO ($v \in \mathcal{Q}_0$), then the characteristic of K is not 2.

2. We can change the definition of $G(\boldsymbol{n},\boldsymbol{g})$ in such a manner that allows us to deal with bilinear forms together with linear mappings. Since bilinear forms on some vector space V are in one to one correspondence with linear mappings from the dual vector space V^* to V, we should change vector spaces assigned to some vertices to the dual ones. In order to do this consider a mapping $\boldsymbol{i} : \mathcal{Q}_0 \to \mathcal{Q}_0$ such that

   c) $\boldsymbol{i}$ is an involution, i.e., $\boldsymbol{i}^2$ is the identical mapping;

   d) $\boldsymbol{n}_{\boldsymbol{i}(v)} = \boldsymbol{n}_v$ for every vertex $v \in \mathcal{Q}_0$.

For every $v \in \mathcal{Q}_0$ with $v < \boldsymbol{i}(v)$ assume that $V_{\boldsymbol{i}(v)} = V_v^*$. Consider the dual basis $e(v,1)^*, \ldots, e(v, \boldsymbol{n}_v)^*$ for V_v^* and identify V_v^* with the space of column vectors of length $\boldsymbol{n}_v$, so $e(v,i)^*$ is the same column vector as $e(v,i)$.

The action of $GL(\boldsymbol{n}_v)$ on V_v induces the action on V_v^*, which we consider as the degree one homogeneous component of the graded algebra $K[V_v]$. Given $g_v \in GL(\boldsymbol{n}_v)$ and $u \in V_v^*$, we have

$$g_v \cdot u = (g_v^{-1})^t u.$$

Hence, we should change the group G to

$$G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}) = \{g \in G(\boldsymbol{n}, \boldsymbol{g}) \mid g_{\boldsymbol{i}(v)} = (g_v^{-1})^t \text{ for all } v \in \mathcal{Q}_0 \text{ with } v < \boldsymbol{i}(v)\}.$$

Since the vector spaces K^n and $(K^n)^*$ are isomorphic as modules over $O(n)$, $Sp(n)$, and $SO(n)$, we assume that

e) if $\boldsymbol{g}_v$ is O, Sp or SO ($v \in \mathcal{Q}_0$), then $\boldsymbol{i}(v) = v$.

3. Instead of the space $H(\mathcal{Q}, \boldsymbol{n})$ we should take its subspace $H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})$, where $\boldsymbol{h} = (\boldsymbol{h}_\alpha)_{\alpha \in \mathcal{Q}_1}$ and $\boldsymbol{h}_\alpha$ is a symbol from the list M, S^+, S^-, L^+, L^-. By definition,

$$H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h}) = \bigoplus_{\alpha \in \mathcal{Q}_1} H_\alpha,$$

where

$$H_\alpha = \begin{cases} K^{\boldsymbol{n}_{\alpha'} \times \boldsymbol{n}_{\alpha''}}, & \text{if } \boldsymbol{h}_\alpha = M \\ S^+(\boldsymbol{n}_{\alpha'}), & \text{if } \boldsymbol{h}_\alpha = S^+ \\ S^-(\boldsymbol{n}_{\alpha'}), & \text{if } \boldsymbol{h}_\alpha = S^- \\ L^+(\boldsymbol{n}_{\alpha'}), & \text{if } \boldsymbol{h}_\alpha = L^+ \\ L^-(\boldsymbol{n}_{\alpha'}), & \text{if } \boldsymbol{h}_\alpha = L^- \end{cases}.$$

Additionally, we have to assume that $\boldsymbol{n}$ and $\boldsymbol{h}$ are subject to the restriction:

f) if $\boldsymbol{h}_\alpha \neq M$ ($\alpha \in \mathcal{Q}_1$), then $\boldsymbol{n}_{\alpha'} = \boldsymbol{n}_{\alpha''}$.

Consider a group $G = G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}) \subset G(\boldsymbol{n})$ and a vector space $H = H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h}) \subset H(\mathcal{Q}, \boldsymbol{n})$ satisfying the previous conditions a)–f). To ensure that these inclusions induce the action of G on H, we assume that the following additional conditions are also valid for all $v \in \mathcal{Q}_0$, $\alpha \in \mathcal{Q}_1$:

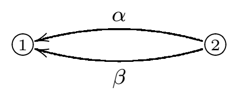
g) if α is a loop, i.e., $\alpha' = \alpha''$, and $\boldsymbol{h}_\alpha$ is S^+ or S^-, then $\boldsymbol{g}_{\alpha'}$ is O or SO;

h) if α is a loop and $\boldsymbol{h}_\alpha$ is L^+ or L^-, then $\boldsymbol{g}_{\alpha'} = Sp$;

i) if α is not a loop and $\boldsymbol{h}_\alpha \neq M$, then $\boldsymbol{i}(\alpha') = \alpha''$ and $\boldsymbol{h}_\alpha$ is S^+ or S^-.

A quintuple $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ satisfying a)–i) is called a *mixed quiver setting*. Definitions of the generic matrices X_α and the algebra of invariants $K[H]^G$ are the same as above. Note that if $\boldsymbol{h}_\alpha = S^+$, then $X_\alpha^t = X_\alpha$; if $\boldsymbol{h}_\alpha = S^-$, then $X_\alpha^t = -X_\alpha$; if $\boldsymbol{h}_\alpha = L^+$, then $(X_\alpha J)^t = X_\alpha J$; and if $\boldsymbol{h}_\alpha = L^-$, then $(X_\alpha J)^t = -X_\alpha J$. In this paper we establish a generating system for $K[H]^G$.

Example 1. 1. Let $\mathcal{Q}$ be the following quiver



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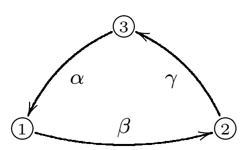
Define a mixed quiver setting $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ by $\boldsymbol{n}_1 = \boldsymbol{n}_2 = n$, $\boldsymbol{g}_1 = \boldsymbol{g}_2 = GL$, $\boldsymbol{h}_\alpha = \boldsymbol{h}_\beta = S^+$, and $\boldsymbol{i}(1) = 2$. The group $G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}) \simeq GL(n)$ acts on $H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h}) = S^+(n) \oplus S^+(n)$ by the rule

$$g \cdot (A, B) = (gAg^t, gBg^t)$$

for $g \in GL(n)$ and $(A, B) \in S^+(n) \oplus S^+(n)$. Hence the orbits of this action correspond to pairs of symmetric bilinear forms on K^n. If we put $\boldsymbol{h}_\alpha = \boldsymbol{h}_\beta = S^-$, then we obtain pairs of skew-symmetric bilinear forms on K^n. The classification problem for such pairs is a classical topic going back to Weierstrass and Kronecker (see [20], [21], and [18]).

2. Let $\mathcal{Q}$ be the following quiver



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Define a mixed quiver setting $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ by $\boldsymbol{n}_1 = \boldsymbol{n}_2 = n$, $\boldsymbol{n}_3 = m$; $\boldsymbol{g}_1 = \boldsymbol{g}_2 = SL$, $\boldsymbol{g}_3 = O$; $\boldsymbol{h}_\alpha = \boldsymbol{h}_\gamma = M$, $\boldsymbol{h}_\beta = S^+$; and $\boldsymbol{i}(1) = 2$, $\boldsymbol{i}(3) = 3$. Hence the action of $G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}) \simeq SL(n) \times O(m)$ on $H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h}) = K^{n\times m} \oplus S^+(n) \oplus K^{m\times n}$ is given by

$$(g, f) \cdot (A, B, C) = (gAf^t, (g^{-1})^t B g^{-1}, fCg^t)$$

for $(g, f) \in SL(n) \times O(m)$ and $(A, B, C) \in K^{n\times m} \oplus S^+(n) \oplus K^{m\times n}$.

3. If we consider mixed quiver settings $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ with the restriction $\boldsymbol{g}_v \in \{GL, O, Sp\}$ for all $v \in \mathcal{Q}_0$, then we obtain the definition of supermixed representations of a quiver (see [35]), or, equivalently, the definition of representations of a signed quiver (see [28]).

2.2 Known results

In this section $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ is a mixed quiver setting, $G = G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i})$, and $H = H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})$. We overview the known results on generators and relations between them for the algebra of invariants $K[H]^G$.

The first results on invariants of quivers (i.e. for the identical involution $\boldsymbol{i}$) were obtained for an important special case of a quiver with one vertex and several loops. For a field of characteristic zero generators for $G \in \{GL(n), SL(n), Sp(n)\}$ were described by Sibirskii in [29] and Procesi in [26]. Procesi also described relations between generators in [26] applying the classical theory of invariants of vectors and covectors (see book [31] by Weyl). Independently, relations for $G = GL(n)$ were described by Razmyslov in [27]. Developing ideas from [26], Aslaksen et al. calculated generators for $G = SO(n)$ (see [2]).

Invariants for an arbitrary quiver $\mathcal{Q}$ were considered by Le Bruyn and Procesi in [22], where generators were found for the case of $G = \prod_{v\in\mathcal{Q}_0} GL(\boldsymbol{n}_n)$ and $H = \sum_{\alpha\in\mathcal{Q}_1} K^{n_{\alpha'}\times n_{\alpha''}}$. Similar results were obtained later by Domokos in [6].

The importance of characteristic-free approach to quiver invariants was pointed out by Formanek in overview [16] (see also [15]). Relying on the theory of modules with good filtrations (see [9]), Donkin described generators for a quiver with one vertex in [11] and for an arbitrary quiver afterwards (see [13]). Relations between generators from the mentioned papers were found by Zubkov in [32] and [34] by means of an approach that allowed to calculate generators and relations between them simultaneously. His method is also based on the theory of modules with good filtrations.

For the rest of classical groups over a field of positive characteristic, the first results were obtained by Zubkov. In [33] he obtained generators for a quiver with one vertex and the orthogonal or symplectic group G. The proof is based on ideas from [11] and a reduction to invariants of mixed quiver settings with $\boldsymbol{g}_v = GL$ for every vertex v. The reduction was performed by means Frobenius reciprocity. Let us recall that we do not consider the case of the (special) orthogonal group in characteristic 2 case. The reason is that in the later case even generators of invariants of several vectors are not known (for the latest developments see [8]).

Invariants of a quiver under the action of $G = \prod_{v\in\mathcal{Q}_0} SL(n_v)$ are called semi-invariants. Its generators for an arbitrary characteristic were established by Domokos and Zubkov in [7] using the methods from [11], [13], [32], [34], and, independently, by Derksen and Weyman in [4], [3] utilizing the methods of the representation theory of quivers. Simultaneously, similar result in the case of characteristic zero was obtained by Schofield and Van den Bergh in [30]. These results were gen-

eralized for mixed quiver settings with $\boldsymbol{g}_v = SL$ and $\boldsymbol{h}_\alpha = M$, where v is a vertex and α is an arrow, by the author and Zubkov in [23].

Zubkov in [35] combined the arguments for Young superclasses from [13] with the reduction from [33] to describe generators for a mixed quiver setting $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ with $\boldsymbol{g}_v \in \{GL, O, Sp\}$ for all $v \in \mathcal{Q}_0$. Relations between them were found in [35].

Generators for the remaining mixed quiver settings are computed in this paper.

3 Preliminaries

3.1 Notations

In what follows, $\mathbb{N}$ stands for the set of non-negative integers, $\mathbb{Z}$ for the set of integers, and $\mathbb{Q}$ for the quotient field of the ring $\mathbb{Z}$.

The cardinality of a set S is denoted by $\#S$ and the permutation group on n elements is denoted by $\mathcal{S}_n$. Given integers $i < j$, we write $[i, j]$ for the interval $i, i+1, \ldots, j-1, j$.

By a *distribution* $B = (B_1, \ldots, B_s)$ of a set $[1, t]$ we mean an ordered partition of the set into pairwise disjoint subsets B_i ($1 \leq i \leq s$), which are called components of the distribution. To every B we associate two functions $j \mapsto B|j|$ and $j \mapsto B\langle j\rangle$ ($1 \leq j \leq t$), defined by the rules:

$$B|j| = i, \text{ if } j \in B_i, \text{ and } B\langle j\rangle = \#\{[1, j] \cap B_i \,|\, j \in B_i\}.$$

A vector $\underline{t} = (t_1, \ldots, t_s) \in \mathbb{N}^s$ determines the distribution $T = (T_1, \ldots, T_s)$ of the set $[1, t]$, where $t = t_1 + \cdots + t_s$ and $T_i = \{t_1 + \cdots + t_{i-1} + 1, \ldots, t_1 + \cdots + t_i\}$, $1 \leq i \leq s$. As an example, if $\underline{t} = (1, 3, 0, 2)$, then $T = (\{1\}, \{2, 3, 4\}, \emptyset, \{5, 6\})$ and $T|5| = 4$.

A vector $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{N}^s$ satisfying $\lambda_1 \geq \cdots \geq \lambda_s$ and $\lambda_1 + \cdots + \lambda_s = t$ is called a *partition* of t and is denoted by $\underline{\lambda} \vdash t$. A *multi-partition* $\underline{\lambda} \vdash \underline{t}$ is a q-tuple of partitions $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_q)$, where $\underline{\lambda}_1 \vdash t_1, \ldots, \underline{\lambda}_q \vdash t_q$, and $\underline{t} = (t_1, \ldots, t_q) \in \mathbb{N}^q$.

3.2 Pfaffians and tableaux with substitutions

Denote coefficients in the characteristic polynomial of an $n \times n$ matrix X by $\sigma_k(X)$, i.e.,

$$\det(\lambda E - X) = \lambda^n - \sigma_1(X)\lambda^{n-1} + \cdots + (-1)^n \sigma_n(X).$$

Assume n is even. Define the *generalized pfaffian* of an arbitrary $n\times n$ matrix $X=(x_{ij})$ by
$$\overline{\mathrm{pf}}(X)=\mathrm{pf}(X-X^t),$$
where pf stands for the pfaffian of a skew-symmetric matrix. By abuse of notation we will refer to $\overline{\mathrm{pf}}$ as the pfaffian. For $K=\mathbb{Q}$ there is a more convenient formula

$$\overline{\mathrm{pf}}(X)=\mathrm{pf}(X-X^t)=\frac{1}{(n/2)!}\sum_{\pi\in\mathcal{S}_n}\mathrm{sgn}(\pi)\prod_{i=1}^{n/2}x_{\pi(2i-1),\pi(2i)}. \tag{1}$$

For $n\times n$ matrices $X_1=(x_{ij}(1)),\ldots,X_s=(x_{ij}(s))$ and positive integers $k_1,\ldots,k_s$, satisfying $k_1+\cdots+k_s=n/2$, consider the polynomial $\overline{\mathrm{pf}}(x_1X_1+\cdots+x_sX_s)$ in the variables $x_1,\ldots,x_s$. The partial linearization $\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(X_1,\ldots,X_s)$ of the pfaffian is the coefficient at $x_1^{k_1}\cdots x_s^{k_s}$ in this polynomial. Assume that for some $\underline{n}=(n_1,\ldots,n_m)\in\mathbb{N}^m$ with $n_1+\cdots+n_m=n$ each of matrices $X_1,\ldots,X_s$ is partitioned into $m\times m$ number of blocks, where the block in the (i,j)-th position is an $n_i\times n_j$ matrix and the only non-zero block is the one in the (p,q)-th position. Then $\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(X_1,\ldots,X_s)$ is called a *block partial linearization of the pfaffian* (*b.p.l.p.*).

The following notions were introduced in Section 3 of [24], where more detailed explanation and examples are given.

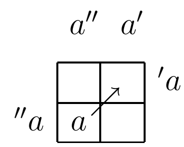
Definition (of shapes). The *shape* of dimension $\underline{n}=(n_1,\ldots,n_m)\in\mathbb{N}^m$ is the collection of m columns of cells. The columns are numbered by $1,2,\ldots,m$, and the i-th column contains exactly n_i cells, where $1\le i\le m$. Numbers $1,\ldots,n_i$ are assigned to the cells of the i-th column, starting from the top. As an example, the shape of dimension $\underline{n}=(3,2,3,1,1)$ is

1	2	3	4	5
1	1	1	1	1
2	2	2		
3		3		

Definition (of a tableau with substitution). Let $\underline{n}=(n_1,\ldots,n_m)\in\mathbb{N}^m$ and let $n=n_1+\cdots+n_m$ be even. A pair $(T,(X_1,\ldots,X_s))$ is called a *tableau with substitution* of dimension $\underline{n}$ if

- T is the shape of dimension $\underline{n}$ together with a set of arrows. An *arrow* goes from one cell of the shape into another one, and each cell of the shape is either

the head or the tail of one and only one arrow. We refer to T as *tableau* of dimension $\underline{n}$, and we write $a \in T$ for an arrow a from T. Given an arrow $a \in T$, denote by a' and a'' the columns containing the head and the tail of a, respectively. Similarly, denote by $'a$ the number assigned to the cell containing the head of a, and denote by $''a$ the number assigned to the cell containing the tail of a. Schematically this is depicted as

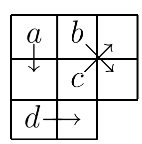


- φ is a fixed mapping from the set of arrows of T onto $[1, s]$ that satisfies the following property:

  if $a, b \in T$ and $\varphi(a) = \varphi(b)$, then $a' = b'$, $a'' = b''$;

- $(X_1, \ldots, X_s)$ is a sequence of matrices such that the matrix $X_{\varphi(a)}$ assigned to the arrow $a \in T$ is $n_{a''} \times n_{a'}$ matrix and its (p, q)-th entry is denoted by $(X_j)_{pq}$.

Example 2. Let T be the tableau



of dimension $(3, 3, 2)$. Define φ by $\varphi(a) = 1$, $\varphi(b) = \varphi(c) = 2$, and $\varphi(d) = 3$, and let X_1, X_3 be 3×3 matrices and X_2 be a 3×2 matrix. Then $(T, (X_1, X_2, X_3))$ is a tableau with substitution.

Definition (of $\mathrm{bpf}_T(X_1, \ldots, X_s)$). Let $(T, (X_1, \ldots, X_s))$ be a tableau with substitution of dimension $\underline{n}$. Define the polynomial

$$\mathrm{bpf}_T^0(X_1, \ldots, X_s) = \sum_{\pi_1 \in \mathcal{S}_{n_1}, \ldots, \pi_m \in \mathcal{S}_{n_m}} \mathrm{sgn}(\pi_1) \cdots \mathrm{sgn}(\pi_m) \prod_{a \in T} (X_{\varphi(a)})_{\pi_{a''}(''a), \pi_{a'}('a)}, \quad (2)$$

and the coefficient

$$c_T = \prod_{j=1}^{s} \#\{a \in T \mid \varphi(a) = j\}!$$

In the case $K = \mathbb{Q}$ define

$$\mathrm{bpf}_T(X_1, \ldots, X_s) = \frac{1}{c_T}\mathrm{bpf}^0_T(X_1, \ldots, X_s).$$

Since $\mathrm{bpf}_T(X_1, \ldots, X_s)$ is a polynomial in entries of $X_1, \ldots, X_s$ with integer coefficients, the definition of $\mathrm{bpf}_T(X_1, \ldots, X_s)$ extends over an arbitrary field.

Example 3. For every $n \times n$ matrix X there is a tableau with substitution (T, X) such that $\det(X) = \mathrm{bpf}_T(X)$. If n is odd, then the same is valid for $\overline{\mathrm{pf}}(X)$ (see Example 2 of [24] for details).

The next lemma, which is part b) of Lemma 1 from [24], shows that bpf is a b.p.l.p.

Lemma 1 *Let $(T, (X_1, \ldots, X_s))$ be a tableau with substitution of dimension $\underline{n} = (n_1, \ldots, n_m)$. Consider $a_1, \ldots, a_s \in T$ such that $\varphi(a_1) = 1, \ldots, \varphi(s) = s$. For any $1 \le p \le s$ denote by Z_p the $n \times n$ matrix, partitioned into $m \times m$ number of blocks, where the block in the (i, j)-th position is an $n_i \times n_j$ matrix; the block in the (a''_p, a'_p)-th position is equal to X_p, and the rest of blocks are zero matrices. Then*

$$\mathrm{bpf}_T(X_1, \ldots, X_s) = \pm\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(Z_1, \ldots, Z_s),$$

where $k_p = \#\{a \in T \,|\, \varphi(a) = p\}$ for any $1 \le p \le s$.

4 Generators

4.1 Main results

Let $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting and $\mathcal{Q}_0 = \{1, \ldots, l\}$. Without loss of generality we can assume that

$$\text{if } v \in \mathcal{Q}_0 \text{ and } \boldsymbol{g}_v \text{ is } GL \text{ or } SL, \text{ then } \boldsymbol{i}(v) \ne v. \tag{3}$$

Otherwise we can add a new vertex $\overline{v}$ to $\mathcal{Q}$, and set $\boldsymbol{i}(v) = \overline{v}$, $\boldsymbol{n}_{\overline{v}} = \boldsymbol{n}_v$, $\boldsymbol{g}_{\overline{v}} = \boldsymbol{g}_v$; this construction changes neither the space $H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})$ nor the algebra of invariants.

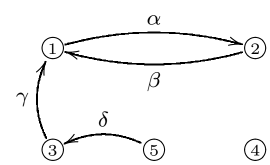
Definition (of the mixed double quiver setting $\mathfrak{Q}^D$). Define the mixed *double* quiver setting $\mathfrak{Q}^D = (\mathcal{Q}^D, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}^D, \boldsymbol{i})$ as follows: $\mathcal{Q}^D_0 = \mathcal{Q}_0$, $\mathcal{Q}^D_1 = \mathcal{Q}_1 \coprod \{\alpha^t \,|\, \alpha \in \mathcal{Q}_1,\ \boldsymbol{h}_\alpha = M\}$, where $(\alpha^t)' = \boldsymbol{i}(\alpha'')$, $(\alpha^t)'' = \boldsymbol{i}(\alpha')$, and $\boldsymbol{h}^D_{\alpha^t} = M$ for $\alpha \in \mathcal{Q}_1$ with $\boldsymbol{h}_\alpha = M$ and $\boldsymbol{h}^D_\alpha = \boldsymbol{h}_\alpha$ for all $\alpha \in \mathcal{Q}_1$.

Define a mapping $\Phi^D : K[H(\mathcal{Q}^D, \boldsymbol{n}, \boldsymbol{h}^D)] \to K[H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})]$ such that $\Phi^D(X_\alpha) = X_\alpha$ for $\alpha \in \mathcal{Q}_1$, and $\Phi^D(X_{\alpha^t})$ for $\alpha \in \mathcal{Q}_1$ and $\boldsymbol{h}_\alpha = M$ is defined as follows:

- If $\boldsymbol{g}_{\alpha'} \neq Sp$ and $\boldsymbol{g}_{\alpha''} \neq Sp$, then $\Phi^D(X_{\alpha^t}) = X_\alpha^t$.
- If $\boldsymbol{g}_{\alpha'} = Sp$ and $\boldsymbol{g}_{\alpha''} \neq Sp$, then $\Phi^D(X_{\alpha^t}) = X_\alpha^t J(\boldsymbol{n}_{\alpha'})$.
- If $\boldsymbol{g}_{\alpha'} \neq Sp$ and $\boldsymbol{g}_{\alpha''} = Sp$, then $\Phi^D(X_{\alpha^t}) = J(\boldsymbol{n}_{\alpha''}) X_\alpha^t$.
- If $\boldsymbol{g}_{\alpha'} = Sp$ and $\boldsymbol{g}_{\alpha''} = Sp$, then $\Phi^D(X_{\alpha^t}) = J(\boldsymbol{n}_{\alpha''}) X_\alpha^t J(\boldsymbol{n}_{\alpha'})$.

Let us remark that the meaning of notation $\Phi^D(X_\alpha)$ was explained in Section 2.1.

Example 4. Let $\mathcal{Q}$ be



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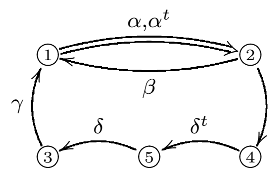
Define a mixed quiver setting $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ by $\boldsymbol{i}(1) = 2$, $\boldsymbol{i}(3) = 4$, $\boldsymbol{i}(5) = 5$; $\boldsymbol{g}_1 = \boldsymbol{g}_2 = GL$, $\boldsymbol{g}_3 = \boldsymbol{g}_4 = SL$, $\boldsymbol{g}_5 = O$; $\boldsymbol{h}_\alpha = \boldsymbol{h}_\gamma = \boldsymbol{h}_\delta = M$, $\boldsymbol{h}_\beta = S^+$. Then $\mathcal{Q}^D$ is



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Before presenting the next concept, let us recall that $\alpha = \alpha_1 \cdots \alpha_r$ is a *path* in $\mathcal{Q}$ (where $\alpha_1, \ldots, \alpha_r \in \mathcal{Q}_1$), if $\alpha_1' = \alpha_2'', \ldots, \alpha_{r-1}' = \alpha_r''$. The head of the path α is $\alpha' = \alpha_r'$ and the tail is $\alpha'' = \alpha_1''$. The path α is called *closed* if $\alpha' = \alpha''$.

Definition (of a $\mathfrak{Q}$-tableau with substitution and a path $\mathfrak{Q}$-tableau with substitution). A tableau with substitution $(T, (Y_1, \ldots, Y_s))$ of dimension $\underline{n} \in \mathbb{N}^m$ is called a $\mathfrak{Q}$-*tableau with substitution*, if for some *weight* $\underline{w} = (w_1, \ldots, w_l) \in \mathbb{N}^l$ and the distribution W, determined by $\underline{w}$ (see Section 3.1), we have

- $\underline{n} = (\underbrace{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_1}_{w_1}, \ldots, \underbrace{\boldsymbol{n}_l \ldots, \boldsymbol{n}_l}_{w_l})$;
- if $a \in T$, then there exists an $\alpha \in \mathcal{Q}_1$ such that $Y_{\varphi(a)} = X_\alpha$, $W|a'| = \boldsymbol{i}(\alpha'')$, $W|a''| = \alpha'$.

If we replace the last condition by the following one

- if $a \in T$, then there exists a path $\alpha = \alpha_1 \cdots \alpha_r$ in $\mathcal{Q}$ (where $\alpha_1, \ldots, \alpha_r \in \mathcal{Q}_1$) such that $Y_{\varphi(a)} = X_{\alpha_r} \cdots X_{\alpha_1}$, $W|a'| = \boldsymbol{i}(\alpha'')$, $W|a''| = \alpha'$,

then we obtain the definition of a *path $\mathfrak{Q}$-tableau with substitution*. Obviously, for a (path) $\mathfrak{Q}$-tableau with substitution $(T, (Y_1, \ldots, Y_s))$ we have $\mathrm{bpf}_T(Y_1, \ldots, Y_s) \in K[H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})]$.

Theorem 1 ***(Main theorem)*** *Let $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting satisfying (3). Then the algebra of invariants $K[H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})]^{G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})}$ is generated as K-algebra by the elements $\Phi^D(\sigma_k(X_{\beta_r} \cdots X_{\beta_1}))$, $\Phi^D(\mathrm{bpf}_T(Y_1, \ldots, Y_s))$, where*

1. *$\beta_1 \cdots \beta_r$ ranges over all closed paths in $\mathcal{Q}^D$ and $1 \leq k \leq \boldsymbol{n}_{\beta_1''}$;*
2. *$(T, (Y_1, \ldots, Y_s))$ ranges over all path $\mathfrak{Q}^D$-tableaux with substitutions of a weight $\underline{w}$ such that*
   a) if $\boldsymbol{g}_v \in \{GL, O, Sp\}$ for some $v \in \mathcal{Q}_0$, then $w_{\boldsymbol{i}(v)} = w_v = 0$;
   b) if $\boldsymbol{g}_v = SL$ for some $v \in \mathcal{Q}_0$, then $w_{\boldsymbol{i}(v)} = 0$ or $w_v = 0$;
   c) if $\boldsymbol{g}_v = SO$ for some $v \in \mathcal{Q}_0$, then $w_v \leq 1$ and $\boldsymbol{i}(v) = v$.

This theorem implies the main result of [35].

Corollary 1 *Let $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting satisfying (3). If $\boldsymbol{g}_v \in \{GL, O, Sp\}$ for all $v \in \mathcal{Q}_0$, then K-algebra $K[H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})]^{G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})}$ is generated by $\Phi^D(\sigma_k(X_{\beta_r} \cdots X_{\beta_1}))$, where $\beta_1 \cdots \beta_r$ is a closed path in $\mathcal{Q}^D$ and $1 \leq k \leq \boldsymbol{n}_{\beta_1''}$.*

For $f \in K[H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h})]$ let $\underline{t} = (t_\alpha)_{\alpha \in \mathcal{Q}_1} \in \mathbb{N}^{\#\mathcal{Q}_1}$ be the *multidegree* of f, i.e., t_α is the total degree of the polynomial f in x_{ij}^α, where $1 \leq i \leq \boldsymbol{n}_{\alpha'}$ and $1 \leq j \leq \boldsymbol{n}_{\alpha''}$. The algebra of invariants is homogeneous with respect to the grading by multidegrees as well as the generating system from Theorem 1.

4.2 Invariants of several matrices

Consider the case of a quiver with one vertex and d loops. Let $\boldsymbol{h}_\alpha = M$ for every arrow α. Then $H = K^{n \times n} \oplus \cdots \oplus K^{n \times n}$ is d-tuple of $n \times n$ matrices over K, and G is a group from the list $GL(n)$, $O(n)$, $Sp(n)$, $SO(n)$. We assume that if G is $O(n)$ or $SO(n)$, then the characteristic of K is nor 2; if G is $Sp(n)$, then n is even. The group G acts on H by the diagonal conjugation. Hence it acts on $K[H]$ as follows: $g \cdot X_\alpha = g^{-1} X_\alpha g$, where $g \in G$, $1 \leq \alpha \leq d$, and $X_\alpha = (x_{ij}^\alpha)$ is the $n \times n$ generic matrix. We do not consider the case $G = SL(n)$ because invariants for $GL(n)$ and $SL(n)$ are the same.

Corollary 2 *The algebra of invariants $K[H]^G$ is generated by the following elements:*

a) $\sigma_k(X)$ *(*$1 \leq k \leq n$ *and* X *ranges over all monomials in* $X_1, \ldots, X_d$*), if* $G = GL(n)$*;*

b) $\sigma_k(Y)$ *(*$1 \leq k \leq n$*), if* $G = O(n)$*;*

c) $\sigma_k(Y)$ *(*$1 \leq k \leq n$*), if* $G = SO(n)$ *and* n *is odd;*

d) $\sigma_k(Y)$, $\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(Y_1, \ldots, Y_s)$ *(*$1 \leq k \leq n$, $k_1 + \cdots + k_s = n/2$*), if* $G = SO(n)$ *and* n *is even.*

In b), c), and d) matrices $Y, Y_1, \ldots, Y_s$ *range over all monomials in* $X_1, \ldots, X_d$, $X_1^t, \ldots, X_d^t$.

e) $\sigma_k(Z)$ *(*$1 \leq k \leq n$ *and* Z *ranges over all monomials in* $X_1, \ldots, X_d$, $JX_1^tJ, \ldots, JX_d^tJ$*), if* $G = Sp(n)$.

Proof. It follows immediately from Theorem 1. To prove part d) we should also use the fact that $\mathrm{bpf}_T(Y_1, \ldots, Y_s)$ is a partial linearization of the pfaffian for one column tableau T (see part 1 of Example 2 from [24]). $\Box$

The only new part of Corollary 2 is part d) (see Section 2.2 for references).

5 Action of groups on tableaux with substitutions

In this section we show that the elements from Theorem 1 are invariants.

Consider an $\underline{n} \in \mathbb{N}^m$ and the group $G = \prod_{i=1}^m GL(n_i)$. Given $g = (g_i)_{1 \leq i \leq m} \in G$ and a tableau with substitution $(T, (Y_1, \ldots, Y_s))$ of dimension $\underline{n}$, define the tableau with substitution $(T, (g * Y_1, \ldots, g * Y_s))$ by $g * Y_j = g_{a''} Y_j g_{a'}^t$, where $1 \leq j \leq s$ and $a \in T$ is such that $\varphi(a) = j$. Since for all $a, b \in T$ with $\varphi(a) = \varphi(b)$ we have $a' = b'$ and $a'' = b''$, the matrix $g * Y_j$ is well defined.

Lemma 2 *Using the preceding notation we have the equality*

$$\mathrm{bpf}_T(g * Y_1, \ldots, g * Y_s) = \det(g_1) \cdots \det(g_m) \mathrm{bpf}_T(Y_1, \ldots, Y_s).$$

Proof. Let $n = n_1 + \cdots + n_m$ and let g_0 be an $n \times n$ block-diagonal matrix such that the i-th block is equal to g_i ($1 \le i \le m$). By the definition of T, n is even and $\{\varphi(a) \,|\, a \in T\} = [1, s]$.

Repeat construction from Lemma 1. Consider $a_1, \ldots, a_s \in T$ such that $\varphi(a_1) = 1, \ldots, \varphi(s) = s$. For any $1 \le p \le s$ denote by Z_p the $n \times n$ matrix, partitioned into $m \times m$ number of blocks, where the block in the (i, j)-th position is an $n_i \times n_j$ matrix; the block in the (a_p'', a_p')-th position is equal to Y_p, and the rest of blocks are zero matrices. Then $\mathrm{bpf}_T(Y_1, \ldots, Y_s) = q\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(Z_1, \ldots, Z_s)$, where $q = \pm 1$, $k_p = \#\{a \in T \,|\, \varphi(a) = p\}$ for any $1 \le p \le s$. Thus,

$$\mathrm{bpf}_T(g * Y_1, \ldots, g * Y_s) = q\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(g_0 Z_1 g_0^t, \ldots, g_0 Z_s g_0^t)$$

$$= q \det(g_0)\overline{\mathrm{pf}}_{k_1,\ldots,k_s}(Z_1, \ldots, Z_s),$$

since $\overline{\mathrm{pf}}(g_0 Z g_0^t) = \det(g_0)\overline{\mathrm{pf}}(Z)$ for every $n \times n$ matrix Z. This completes the proof. $\square$

Lemma 3 *Let $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting that satisfies (3) and $\mathcal{Q}_0 = \{1, \ldots, l\}$. Let $(T, (Y_1, \ldots, Y_s))$ be a $\mathfrak{Q}$-tableau with substitution or path $\mathfrak{Q}$-tableau with substitution of a weight $\underline{w} = (w_1, \ldots, w_l)$. Assume that $w_v = 0$ for all $v \in \mathcal{Q}_0$ with $\boldsymbol{g}_v = Sp$. Then for all $g = (g_v)_{v \in \mathcal{Q}_0} \in G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i})$ we have*

$$g \cdot \mathrm{bpf}_T(Y_1, \ldots, Y_s) = q\, \mathrm{bpf}_T(Y_1, \ldots, Y_s),$$

where

$$q = \prod_{v \in \mathcal{Q}_0,\, v = \boldsymbol{i}(v)} \det(g_v)^{-w_v} \prod_{v \in \mathcal{Q}_0,\, v < \boldsymbol{i}(v)} \det(g_v)^{w_{\boldsymbol{i}(v)} - w_v}.$$

Proof. **a)** Assume that $(T, (Y_1, \ldots, Y_s))$ is a $\mathfrak{Q}$-tableau with substitution. Let $w = w_1 + \cdots + w_l$ and let W be the distribution determined by $\underline{w}$. Define a mapping $\psi : G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}) \to \prod_{j=1}^{w} GL(\boldsymbol{n}_{W|j|})$ by $\psi(g) = (\psi(g)_1, \ldots, \psi(g)_w)$ and $\psi(g)_j = g_{W|j|}$ for $1 \le j \le w$. We claim that

$$g \cdot \mathrm{bpf}_T(Y_1, \ldots, Y_s) = \mathrm{bpf}_T(\psi(g^{-1}) * Y_1, \ldots, \psi(g^{-1}) * Y_s) \text{ for any } g \in G(\boldsymbol{n}, \boldsymbol{g}, \boldsymbol{i}). \quad (4)$$

By the definition of a $\mathfrak{Q}$-tableau with substitution, for every $a \in T$ there exists an arrow $\alpha \in \mathcal{Q}_1$ with $Y_{\varphi(a)} = X_\alpha$, $W|a'| = \boldsymbol{i}(\alpha'')$, and $W|a''| = \alpha'$. Hence $g \cdot Y_{\varphi(a)} = g \cdot X_\alpha = g_{\alpha'}^{-1} X_\alpha g_{\alpha''}$. On the other hand, $\psi(g^{-1}) * Y_{\varphi(a)} = \psi(g^{-1})_{a''} X_\alpha \psi(g^{-1})_{a'}^t =$

$g^{-1}_{W|a''|}X_\alpha(g^{-1}_{W|a'|})^t = g^{-1}_{\alpha'}X_\alpha(g^{-1}_{\boldsymbol{i}(\alpha'')})^t = g^{-1}_{\alpha'}X_\alpha g_{\alpha''}$, since $g_{\boldsymbol{i}(\alpha'')} = (g^{-1}_{\alpha''})^t$ for $\boldsymbol{g}_{\alpha''} \neq Sp$. Therefore $g \cdot Y_{\varphi(a)} = \psi(g^{-1}) * Y_{\varphi(a)}$ and (4) is proven.

Equality (4) together with Lemma 2 completes the proof.

b) Let $(T,(Y_1,\ldots,Y_s))$ be a path $\mathfrak{Q}$-tableau with substitution. Observe that for any $g \in G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$ and a path $\alpha = \alpha_1\cdots\alpha_r$ in $\mathcal{Q}$ we have $g \cdot X_{\alpha_r}\cdots X_{\alpha_1} = g^{-1}_{\alpha'} \cdot X_{\alpha_r}\cdots X_{\alpha_1} \cdot g_{\alpha''}$. Use this remark and the proof of part a) to obtain the claim. $\square$

Lemma 4 *The elements from Theorem 1 are invariants.*

Proof. We claim that $g \cdot \Phi^D(X_\beta) = \Phi^D(g \cdot X_\beta)$ for all $g \in G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$ and $\beta \in \mathcal{Q}^D_1$. For $v \in \mathcal{Q}_0$ set

$$I_v = \left\{ \begin{array}{ll} J(\boldsymbol{n}_v), & \text{if } \boldsymbol{g}_v = Sp \\ E(\boldsymbol{n}_v), & \text{otherwise} \end{array} \right., \quad \delta_v = \left\{ \begin{array}{ll} -1, & \text{if } \boldsymbol{g}_v = Sp \\ 1, & \text{otherwise} \end{array} \right. .$$

Then for every $v \in \mathcal{Q}_0$ we have

$$\boldsymbol{g}_{\boldsymbol{i}(v)} = \delta_v I_v (g_v^{-1})^t I_v. \tag{5}$$

There are two cases.

- If $\beta \in \mathcal{Q}_1$, then $g \cdot \Phi^D(X_\beta) = g \cdot X_\beta = \Phi^D(g \cdot X_\beta)$.
- If $\beta = \alpha^t$ for $\alpha \in \mathcal{Q}_1$, then $g \cdot \Phi^D(X_\beta) = g \cdot (I_{\alpha''}X^t_\alpha I_{\alpha'}) = I_{\alpha''}(g^{-1}_{\alpha'}X_\alpha g_{\alpha''})^t I_{\alpha'}$. On the other hand, $\Phi^D(g \cdot X_\beta) = \Phi^D(g^{-1}_{\beta'}X_\beta g_{\beta''}) = g^{-1}_{\boldsymbol{i}(\alpha'')}I_{\alpha''}X^t_\alpha I_{\alpha'} g_{\boldsymbol{i}(\alpha')}$. Formula (5) shows that $g \cdot \Phi^D(X_\beta) = \Phi^D(g \cdot X_\beta)$.

Therefore $g \cdot \Phi^D(X_\beta) = \Phi^D(g \cdot X_\beta)$ and, consequently, $g \cdot \Phi^D(f) = \Phi^D(g \cdot f)$ for every $f \in K[H(\mathcal{Q}^D,\boldsymbol{n},\boldsymbol{h}^D)]$. Obviously, for a closed path $\beta_1\cdots\beta_r$ in $\mathcal{Q}^D$ and $1 \leq k \leq \boldsymbol{n}_{\beta_1''}$, the element $\sigma_k(X_{\beta_r}\cdots X_{\beta_1}) \in K[H(\mathcal{Q}^D,\boldsymbol{n},\boldsymbol{h}^D)]$ is a $G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$-invariant. This remark together with Lemma 3 concludes the proof. $\square$

6 GL- and SL-invariants of quivers

Let $\mathfrak{Q} = (\mathcal{Q},\boldsymbol{n},\boldsymbol{g},\boldsymbol{h},\boldsymbol{i})$ be a mixed quiver setting satisfying (3) and $G = G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$, $H = H(\mathcal{Q},\boldsymbol{n},\boldsymbol{h})$. Throughout this section we assume that $\boldsymbol{g}_v$ is GL or SL for all $v \in \mathcal{Q}_0$ and $\boldsymbol{h}_\alpha = M$ for all $\alpha \in \mathcal{Q}_1$.

Definition (of $\mathfrak{Q}^L$). Define a mixed quiver setting $\mathfrak{Q}^L = (\mathcal{Q}^L, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}^L, \boldsymbol{i})$ together with a mapping $\Phi^L : K[H(\mathcal{Q}^L, \boldsymbol{n}, \boldsymbol{h}^L)] \to K[H]$ as follows. (Here the letter L stands for the word *loop*). Let $\mathcal{Q}_0^L = \mathcal{Q}_0$, $\mathcal{Q}_1^L = \mathcal{Q}_1 \coprod \{\alpha_v \,|\, v \in \mathcal{Q}_0,\, v < \boldsymbol{i}(v)\}$, where α_v is a loop in the vertex v. For an $\alpha \in \mathcal{Q}_1$ define $\boldsymbol{h}_\alpha^L = \boldsymbol{h}_\alpha$ and $\Phi^L(X_\alpha) = X_\alpha$; for $v \in \mathcal{Q}_0$ with $v < \boldsymbol{i}(v)$ define $\boldsymbol{h}_{\alpha_v}^L = M$ and $\Phi^L(X_{\alpha_v}) = E(\boldsymbol{n}_v)$.

It is not difficult to see that $g \cdot \Phi^L(X_\beta) = \Phi^L(g \cdot X_\beta)$ for all $g \in G$ and $\beta \in \mathcal{Q}_1^L$. Therefore

$$g \cdot \Phi^L(f) = \Phi^L(g \cdot f) \text{ for all } f \in K[H(\mathcal{Q}^L, \boldsymbol{n}, \boldsymbol{h}^L)] \tag{6}$$

Theorem 2 *Let $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting satisfying (3), where $\boldsymbol{g}_v$ is GL or SL for all $v \in \mathcal{Q}_0$ and $\boldsymbol{h}_\alpha = M$ for all $\alpha \in \mathcal{Q}_1$. Then the algebra of invariants $K[H]^G$ is spanned over K by the elements $\Phi^L(\mathrm{bpf}_T(Y_1, \ldots, Y_s))$, where $(T, (Y_1, \ldots, Y_s))$ is a $\mathfrak{Q}^L$-tableau with substitution of a weight $\underline{w}$ and $w_{\boldsymbol{i}(v)} = w_v$ for all $v \in \mathcal{Q}_0$ with $\boldsymbol{g}_v = GL$.*

The proof is organized as follows. The semi-invariants, i.e., the invariants for the case of $\boldsymbol{g}_v = SL$ for all $v \in \mathcal{Q}_0$, has been calculated in [23], where the case of an arbitrary quiver setting was reduced to a *zigzag* quiver setting, and for zigzag quiver settings generating systems for semi-invariants were described. We start by rewriting these results in the language of tableaux with substitutions (see Sections 6.1, 6.2). Afterwards it is just an exercise to reduce GL- and SL-invariants to semi-invariants (see Section 6.3).

6.1 Semi-invariants of zigzag quiver settings

The following two definitions are taken from [23].

Definition. A quiver $\mathcal{Q}$ is called *bipartite*, if every vertex is a source (i.e. there is no arrow ending at this vertex), or a sink (i.e. there is no arrow starting at this vertex). A quiver setting $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ is called a *zigzag* quiver setting, if

- $\mathcal{Q}$ is a bipartite quiver, $\boldsymbol{g}_v$ is SL for all $v \in \mathcal{Q}_0$ and $\boldsymbol{h}_\alpha = M$ for all $\alpha \in \mathcal{Q}_1$;
- for every vertex $v \in \mathcal{Q}_0$ we have $\boldsymbol{i}(v) \neq v$; moreover, if v is a source, then $\boldsymbol{i}(v)$ is a sink and vice versa;
- there is no arrow $\alpha \in \mathcal{Q}_1$ with $\alpha' > \boldsymbol{i}(\alpha')$ and $\alpha'' > \boldsymbol{i}(\alpha'')$.

Assume that $(\mathcal{Q},\boldsymbol{n},\boldsymbol{g},\boldsymbol{h},\boldsymbol{i})$ is a zigzag quiver setting. Hence $\mathcal{Q}$ can be schematically depicted as follows.

$$\begin{array}{rcccl}
l_1+l_2+1 & \bullet & & \bullet & 1\\
& \vdots & \stackrel{\beta(1),\ldots,\beta(d_2)}{\longrightarrow} & \vdots & \\
2l_1+l_2 & \bullet & & \bullet & l_1\\
& & \stackrel{\alpha(1),\ldots,\alpha(d_1)}{\nearrow} & & \\
l_1+1 & \bullet & & \bullet & 2l_1+l_2+1\\
& \vdots & \stackrel{\gamma(1),\ldots,\gamma(d_3)}{\longrightarrow} & \vdots & \\
l_1+l_2 & \bullet & & \bullet & 2l_1+2l_2
\end{array}$$

Here

- $\mathcal{Q}_0=\{1,\ldots,2l_1+2l_2\}$ and
  $\mathcal{Q}_1=\{\alpha(1),\ldots,\alpha(d_1),\beta(1),\ldots,\beta(d_2),\gamma(1),\ldots,\gamma(d_3)\}$;
- the arrows $\alpha(1),\ldots,\alpha(d_1)$ go from the vertices $l_1+1,\ldots,l_1+l_2$ to the vertices $1,\ldots,l_1$; the arrows $\beta(1),\ldots,\beta(d_2)$ go from $l_1+l_2+1,\ldots,2l_1+l_2$ to $1,\ldots,l_1$; and the arrows $\gamma(1),\ldots,\gamma(d_3)$ go from $l_1+1,\ldots,l_1+l_2$ to $2l_1+l_2+1,\ldots,2l_1+2l_2$;
- the involution $\boldsymbol{i}$ permutes vertices horizontally; consequently $\boldsymbol{n}_{\boldsymbol{i}(v)}=\boldsymbol{n}_v$ for all $v\in\{1,\ldots,2l_1+2l_2\}$.

Fix $\underline{t}=(t_1,\ldots,t_{d_1})$, $\underline{r}=(r_1,\ldots,r_{d_2})$, $\underline{s}=(s_1,\ldots,s_{d_3})$ and denote by $K[H](\underline{t},\underline{r},\underline{s})\subset K[H]$ the space of polynomials that have a total degree t_k in variables from $X_{\alpha(k)}$ $(1\leq k\leq d_1)$, a total degree r_k in variables from $X_{\beta(k)}$ $(1\leq k\leq d_2)$, and a total degree s_k in variables from $X_{\gamma(k)}$ $(1\leq k\leq d_3)$. Further, let T,R,S, respectively, be the distributions determined by $\underline{t}$, $\underline{r}$, $\underline{s}$, respectively, and denote $t=t_1+\cdots+t_{d_1}$, $r=r_1+\cdots+r_{d_2}$, $s=s_1+\cdots+s_{d_3}$.

Definition. Let $A=(A_1,\ldots,A_p)$ be a distribution of the set $[1,t+2r]$ and let $B=(B_1,\ldots,B_q)$ be a distribution of the set $[1,t+2s]$. The quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ is called *admissible* if there are $\underline{p}=(p_1,\ldots,p_{l_1})\in\mathbb{N}^{l_1}$, $\underline{q}=(q_1,\ldots,q_{l_2})\in\mathbb{N}^{l_2}$ such that for $p=p_1+\cdots+p_{l_1}$, $q=q_1+\cdots+q_{l_2}$ and for the distributions P,Q, determined by $\underline{p}$, $\underline{q}$, we have $\#A_j=\boldsymbol{n}_{P|j|}$ $(1\leq j\leq p)$, $\#B_j=\boldsymbol{n}_{Q|j|+l_1}$ $(1\leq j\leq q)$,

$$\bigcup_{1\leq k\leq d_1,\,\alpha(k)'=i} T_k \bigcup_{1\leq k\leq d_2,\,\beta(k)'=i}(t+R_k) \bigcup_{1\leq k\leq d_2,\,\beta(k)''=\boldsymbol{i}(i)}(t+r+R_k)=\bigcup_{1\leq j\leq p,\,P|j|=i} A_j,$$

where $1 \le i \le l_1$, and

$$\bigcup_{1\le k\le d_1,\,\alpha(k)''=i} T_k \quad \bigcup_{1\le k\le d_3,\,\gamma(k)'=\boldsymbol{i}(i)} (t+S_k) \quad \bigcup_{1\le k\le d_3,\,\gamma(k)''=i} (t+s+S_k) = \bigcup_{1\le j\le q,\,Q|j|=i-l_1} B_j,$$

where $l_1+1 \le i \le l_1+l_2$. In particular, $\sum_{i=1}^{l_1} \boldsymbol{n}_i p_i = t+2r$, $\sum_{i=l_1+1}^{l_1+l_2} \boldsymbol{n}_i q_{i-l_1} = t+2s$. A pair $(\underline{p},\underline{q})$ is called a *weight*.

The definition of the polynomial $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}} \in K[H](\underline{t},\underline{r},\underline{s})$ for an admissible quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ can be found in Section 5.2 of [23]. This polynomial is the block partial linearization of the DP, where DP was introduced in [23] as the mixture of the determinant and the pfaffian. (Note that DP is also a b.p.l.p., see part 4 of Example 2 from [24].) By Proposition 3 from [23], in the case $K=\mathbb{Q}$ we have

$$\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}} = \frac{1}{c} \sum_{\tau_1\in\mathcal{S}_A} \sum_{\tau_2\in\mathcal{S}_B} F^{A,B}(\tau_1,\tau_2), \tag{7}$$

where $F^{A,B}(\tau_1,\tau_2)$ is

$$\mathrm{sgn}(\tau_1)\,\mathrm{sgn}(\tau_2) \prod_{i=1}^{t} x^{T|i|}_{A\langle\tau_1(i)\rangle,B\langle\tau_2(i)\rangle} \prod_{j=1}^{r} y^{R|j|}_{A\langle\tau_1(t+j)\rangle,A\langle\tau_1(t+r+j)\rangle} \prod_{k=1}^{s} z^{S|k|}_{B\langle\tau_2(t+k)\rangle,B\langle\tau_2(t+s+k)\rangle}$$

and the constant $c\in\mathbb{Z}$ depends on the quintuple. Moreover, all coefficients in $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$ belong to $\mathbb{Z}$, if the characteristic of K is zero, and they belong to $\mathbb{Z}/(\mathrm{char}\,K)\mathbb{Z}$, if the characteristic of K is positive. Hence using (7), we can define $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$ over an arbitrary field. Now we can formulate Theorem 2 of [23].

Theorem 3 *If a space $K[H](\underline{t},\underline{r},\underline{s})^G$ is non-zero, then the triplet $(\underline{t},\underline{r},\underline{s})$ is admissible. In this case $K[H](\underline{t},\underline{r},\underline{s})^G$ is spanned over K by the set of all $\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$ for all admissible quintuples $(\underline{t},\underline{r},\underline{s},A,B)$.*

Construction. Let $(\underline{t},\underline{r},\underline{s},A,B)$ be an admissible quintuple of a weight $(\underline{p},\underline{q})$. Define $\underline{n} = (\underbrace{\boldsymbol{n}_1,\ldots,\boldsymbol{n}_1}_{p_1},\ldots,\underbrace{\boldsymbol{n}_{l_1},\ldots,\boldsymbol{n}_{l_1}}_{p_{l_1}},\underbrace{\boldsymbol{n}_{l_1+1},\ldots,\boldsymbol{n}_{l_1+1}}_{q_1},\ldots,\underbrace{\boldsymbol{n}_{l_1+l_2},\ldots,\boldsymbol{n}_{l_1+l_2}}_{q_{l_2}})$. Construct a tableau with substitution $(D,(Z_1,\ldots,Z_h))$ of dimension $\underline{n}$ as follows. Arrows of D are $\{a_i,b_j,c_k\,|\,1\le i\le t,\ 1\le j\le r,\ 1\le k\le s\}$, where

- $''a_i = A\langle i\rangle$, $a_i'' = A|i|$, $'a_i = B\langle i\rangle$, $a_i' = B|i|+p$;

- $''b_j = A\langle t+j\rangle$, $b''_j = A|t+j|$, $'b_j = A\langle t+r+j\rangle$, $b'_j = A|t+r+j|$;
- $''c_k = B\langle t+k\rangle$, $c''_k = B|t+k|+p$, $'c_k = B\langle t+s+k\rangle$, $c'_k = B|t+s+k|+p$.

Define $\varphi(\cdot)$ in such a way that $\{\varphi(a)\,|\,a\in D\} = [1,h]$ for some $h>0$ and for any $1\leq i_1<i_2\leq t$, $1\leq j_1<j_2\leq r$, $1\leq k_1<k_2\leq s$ we have

- $\varphi(a_{i_1})\leq\varphi(a_{i_2})<\varphi(b_{j_1})\leq\varphi(b_{j_2})<\varphi(c_{k_1})\leq\varphi(c_{k_2})$;
- $\varphi(a_{i_1})=\varphi(a_{i_2})$ if and only if $a'_{i_1}=a'_{i_2}$, $a''_{i_1}=a''_{i_2}$, $T|i_1|=T|i_2|$;
- $\varphi(b_{j_1})=\varphi(b_{j_2})$ if and only if $b'_{j_1}=b'_{j_2}$, $b''_{j_1}=b''_{j_2}$, $R|j_1|=R|j_2|$;
- $\varphi(c_{k_1})=\varphi(c_{k_2})$ if and only if $c'_{k_1}=c'_{k_2}$, $c''_{k_1}=c''_{k_2}$, $S|k_1|=S|k_2|$.

Define matrices $Z_1,\ldots,Z_h$ by

- $Z_{\varphi(a_i)}=X_{\alpha(T|i|)}$, $Z_{\varphi(b_j)}=X_{\beta(R|j|)}$, $Z_{\varphi(c_k)}=X_{\gamma(S|k|)}$

for $1\leq i\leq t$, $1\leq j\leq r$, $1\leq k\leq s$. We say that $(D,(Z_1,\ldots,Z_h))$ is the tableau with substitution that corresponds to $(\underline{t},\underline{r},\underline{s},A,B)$.

Lemma 5 *Let $(D,(Z_1,\ldots,Z_h))$ be the tableau with substitution, of dimension $\underline{n}$, that corresponds to an admissible quintuple $(\underline{t},\underline{r},\underline{s},A,B)$ of a weight $(\underline{p},\underline{q})$. Then*

a) $(D,(Z_1,\ldots,Z_h))$ *is well defined;*

b) $(D,(Z_1,\ldots,Z_h))$ *is a $\mathfrak{Q}$-tableau with substitution with the weight* $\underline{w} = (p_1,\ldots,p_{l_1},\,0,\ldots,0,\,q_1,\ldots,q_{l_2})\in\mathbb{N}^{2l_1+2l_2}$*;*

c) $\mathrm{bpf}_D(Z_1,\ldots,Z_h)=\pm\mathrm{DP}^{A,B}_{\underline{t},\underline{r},\underline{s}}$.

Proof. **a)** We claim that $X_{\alpha(T|i|)}$ is an $n_{a''_i}\times n_{a'_i}$ matrix for $1\leq i\leq t$. In other words, we should prove $n_{a''_i}=\boldsymbol{n}_{\alpha(T|i|)'}$ and $n_{a'_i}=\boldsymbol{n}_{\alpha(T|i|)''}$. Admissibility of $(\underline{t},\underline{r},\underline{s},A,B)$ implies that $P|A|i||=\alpha(T|i|)'$ and $Q|B|i||+l_1=\alpha(T|i|)''$. Thus

$$n_{a''_i}=n_{A|i|}=\boldsymbol{n}_{P|A|i||}=\boldsymbol{n}_{\alpha(T|i|)'}\text{ and}$$

$$n_{a'_i}=n_{B|i|+p}=\boldsymbol{n}_{Q|B|i||+l_1}=\boldsymbol{n}_{\alpha(T|i|)''}.$$

Similarly, we can show that $X_{\beta(R|j|)}$ is an $n_{b''_j}\times n_{b'_j}$ matrix for any $1\leq j\leq r$ and $X_{\gamma(S|k|)}$ is an $n_{c''_k}\times n_{c'_k}$ matrix for any $1\leq k\leq s$.

It is not difficult to see that each cell of D is the head or the tail of one and only one arrow. The claim follows.

b) This part of the lemma is straightforward.

c) Construct multi-partitions $\underline{\gamma}_{max} \vdash \underline{t}$, $\underline{\delta}_{max} \vdash \underline{r}$, $\underline{\lambda}_{max} \vdash \underline{s}$, respectively, using $(\underline{t}, \underline{r}, \underline{s}, A, B)$ (see part (iii) of Definition 3 from [23]) and let they determine the distributions Γ_{max}, Δ_{max}, Λ_{max}, respectively. It is not difficult to see that $c_D = \#\mathcal{S}_{\Gamma_{max}} \#\mathcal{S}_{\Delta_{max}} \#\mathcal{S}_{\Lambda_{max}} = \pm c$. Formula (7) concludes the proof over the field $\mathbb{Q}$ which extends to the case of an arbitrary field. $\square$

The following result is a consequence of Theorem 3 and Lemmas 3, 5.

Theorem 4 *Let $(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a zigzag quiver setting. Then the algebra of invariants $K[H]^G$ is spanned over K by the elements $\mathrm{bpf}_D(Z_1, \ldots, Z_h)$, where $(D, (Z_1, \ldots, Z_h))$ is a $\mathfrak{Q}$-tableau with substitution.*

Let us remark that under the given restrictions on $\mathfrak{Q}$ the generating set of Theorem 4 is smaller than that of Theorem 2.

6.2 Semi-invariants of arbitrary quivers

Consider a mixed quiver setting $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ such that $\mathcal{Q}$ is an arbitrary quiver, $\mathcal{Q}_0 = \{1, \ldots, l\}$, $\boldsymbol{g}_v = SL$ for all $v \in \mathcal{Q}_0$, and $\boldsymbol{h}_\alpha = M$ for all $\alpha \in Q_1$. Recall the construction of the quiver $\mathcal{Q}^{(2)}$ from Section 4 of [23]. By definition, $\mathcal{Q}_0^{(2)} = \{1, \ldots, 2l\} \subset \mathbb{N}$ and $\mathcal{Q}_1^{(2)} = \{\overline{\alpha}, \beta_v \,|\, \alpha \in \mathcal{Q}_1,\, v \in \mathcal{Q}_0, \text{ and } v < \boldsymbol{i}(v)\}$, where $\beta_v' = 2v - 1$, $\beta_v'' = 2v$, and

- if $\boldsymbol{i}(\alpha') \geq \alpha'$ or $\boldsymbol{i}(\alpha'') \geq \alpha''$, then $\overline{\alpha}' = 2\alpha' - 1$ and $\overline{\alpha}'' = 2\alpha''$;
- if $\boldsymbol{i}(\alpha') < \alpha'$ and $\boldsymbol{i}(\alpha'') < \alpha''$, then $\overline{\alpha}' = 2\boldsymbol{i}(\alpha'') - 1$ and $\overline{\alpha}'' = 2\boldsymbol{i}(\alpha')$.

Consider the mixed quiver setting $\mathfrak{Q}^{(2)} = (\mathcal{Q}^{(2)}, \boldsymbol{n}^{(2)}, \boldsymbol{g}^{(2)}, \boldsymbol{h}^{(2)}, \boldsymbol{i}^{(2)})$, where for all $v \in \mathcal{Q}_0$, $u \in \mathcal{Q}_0^{(2)}$, and $\gamma \in \mathcal{Q}_0^{(2)}$ we have

$$\boldsymbol{i}^{(2)}(2v-1) = 2\boldsymbol{i}(v), \quad \boldsymbol{i}^{(2)}(2v) = 2\boldsymbol{i}(v) - 1,$$
$$\boldsymbol{n}^{(2)}_{2v} = \boldsymbol{n}^{(2)}_{2v-1} = \boldsymbol{n}_v,$$
$$\boldsymbol{g}^{(2)}(u) = SL, \quad \boldsymbol{h}^{(2)}(\gamma) = M.$$

Denote $G^{(2)} = G(\boldsymbol{n}^{(2)}, \boldsymbol{g}^{(2)}, \boldsymbol{i}^{(2)})$ and $H^{(2)} = H(\mathcal{Q}^{(2)}, \boldsymbol{n}^{(2)}, \boldsymbol{h}^{(2)})$. Then $\mathfrak{Q}^{(2)}$ is a zigzag quiver setting and a generating system of $K[H^{(2)}]^{G^{(2)}}$ is known. The following is the statement of Theorem 1 from [23].

Theorem 5 *The homomorphism of K-algebras $\Phi : K[H^{(2)}]^{G^{(2)}} \to K[H]^G$, given by*

$$\begin{array}{rcl} \Phi(X_{\beta_v}) & = & E(\boldsymbol{n}_v) \text{ for } v \in \mathcal{Q}_0,\ v < \boldsymbol{i}(v), \\ \Phi(X_{\overline{\alpha}}) & = & \left\{ \begin{array}{ll} X_\alpha^t, & \text{if } \boldsymbol{i}(\alpha') < \alpha' \text{ and } \boldsymbol{i}(\alpha'') < \alpha'' \\ X_\alpha, & \text{otherwise} \end{array} \right. \end{array}$$

for $\alpha \in \mathcal{Q}_1$, is a surjective mapping.

Lemma 6 *For every $\mathfrak{Q}^{(2)}$-tableau with substitution $(D, (Z_1, \ldots, Z_s))$ there is a $\mathfrak{Q}^L$-tableau with substitution $(T, (Y_1, \ldots, Y_s))$ such that $\Phi^L(\mathrm{bpf}_T(Y_1, \ldots, Y_s)) = \Phi(\mathrm{bpf}_D(Z_1, \ldots, Z_s))$.*

Proof. Let $(D, (Z_1, \ldots, Z_s))$ be $\mathfrak{Q}^{(2)}$-tableau with substitution of weight $\underline{w} = (w_1, \ldots, w_{2l})$ and dimension $\underline{n}$ and let W be the distribution determined by $\underline{w}$.

For every $v \in \mathcal{Q}_0$ the vertex $2v$ of $\mathcal{Q}^{(2)}$ is a source and $\boldsymbol{i}^{(2)}(2v)$ is a sink. Assume that $w_{2v} \neq 0$. Hence there is an $a \in D$ such that a' or a'' lies in W_{2v}. The definition of a $\mathfrak{Q}^{(2)}$-tableau with substitution implies a contradiction. Thus $w_{2v} = 0$ for all $v \in \mathcal{Q}_0$.

Let $\underline{u} = (u_1, \ldots, u_l)$, where $u_v = w_{2v-1}$ for all $v \in \mathcal{Q}_0$, and let U be the distribution determined by $\underline{u}$. Note that for any $1 \leq i \leq w_1 + \cdots + w_{2l}$ we have $W|i| = 2U|i| - 1$.

Define a tableau with substitution $(T, (Y_1, \ldots, Y_s))$ of dimension $\underline{n}$ as follows. Define $T = \{b_a \,|\, a \in D\}$ and for $a \in D$ define $\varphi(b_a) = \varphi(a)$.

Consider $\gamma \in \mathcal{Q}_1^{(2)}$ such that $Z_{\varphi(a)} = X_\gamma$ and $W|a'| = \boldsymbol{i}^{(2)}(\gamma'')$, $W|a''| = \gamma'$.

If $\gamma = \overline{\alpha}$ for an $\alpha \in \mathcal{Q}_1$, then define $Y_{\varphi(a)} = X_\alpha$. If $\boldsymbol{i}(\alpha') < \alpha'$ and $\boldsymbol{i}(\alpha'') < \alpha''$, then $U|a'| = \alpha'$, $U|a''| = \boldsymbol{i}(\alpha'')$ and we define ${b_a}'' = a'$, ${b_a}' = a''$, $''b_a = {}'a$, $'b_a = {}''a$. If $\boldsymbol{i}(\alpha') > \alpha'$ or $\boldsymbol{i}(\alpha'') > \alpha''$, then $U|a'| = \boldsymbol{i}(\alpha'')$, $U|a''| = \alpha'$ and we define $b_a = a$.

If $\gamma = \beta_v$ for $v \in \mathcal{Q}_0$, $v < \boldsymbol{i}(v)$, then $U|a'| = \boldsymbol{i}(\alpha_v'')$, $U|a''| = \alpha_v'$ for $\alpha_v \in \mathcal{Q}_0^L$ and we define $Y_{\varphi(a)} = X_{\alpha_v}$, $b_a = a$.

This describes a $\mathfrak{Q}^L$-tableau with substitution $(T, (Y_1, \ldots, Y_s))$ of dimension $\underline{n}$ and weight $\underline{u}$. Obviously, $\Phi^L(\mathrm{bpf}_T(Y_1, \ldots, Y_s)) = \Phi(\mathrm{bpf}_D(Z_1, \ldots, Z_s))$. $\Box$

Theorem 6 *Let $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ be a mixed quiver setting satisfying (3), $\boldsymbol{g}_v = SL$ for all $v \in \mathcal{Q}_0$, and $\boldsymbol{h}_\alpha = M$ for all $\alpha \in \mathcal{Q}_1$. Then the algebra of invariants $K[H]^G$ is spanned over K by the elements $\Phi^L(\mathrm{bpf}_T(Y_1, \ldots, Y_s))$, where $(T, (Y_1, \ldots, Y_s))$ is a $\mathfrak{Q}^L$-tableau with substitution.*

Proof. Theorems 4, 5 together with Lemma 6 show that the invariants belong to K-span of the elements from the theorem. Formula (6) and Lemma 3 completes the proof. $\Box$

6.3 Proof of Theorem 2

Let $\mathcal{Q}_0 = \{1,\ldots,l\}$. Define $\boldsymbol{g}^{(1)} = (\boldsymbol{g}_1^{(1)},\ldots,\boldsymbol{g}_l^{(1)})$ by $\boldsymbol{g}_v^{(1)} = SL$ for all $v \in \mathcal{Q}_0$ and denote $G^{(1)} = G(\boldsymbol{n},\boldsymbol{g}^{(1)},\boldsymbol{i})$. Since $G^{(1)} \subset G$, we have $K[H]^G \subset K[H]^{G^{(1)}}$. Also note that $\mathcal{Q}_0^L = \mathcal{Q}_0$.

Let $f \in K[H]^G$ be a polynomial of a multidegree $\underline{t} = (t_\alpha)_{\alpha\in\mathcal{Q}_1}$. By Theorem 6, $f = \sum_j \lambda_j \Phi^L(\mathrm{bpf}_{T_j}(Y_{j,1},\ldots,Y_{j,s_j}))$, where $\lambda_j \in K$, $(T_j,(Y_{j,1},\ldots,Y_{j,s_j}))$ is a $\mathfrak{Q}^L$-tableau with substitution of a weight $\underline{w}_j = (w_{j,1},\ldots,w_{j,l})$, and the multidegree of $\Phi^L(\mathrm{bpf}_{T_j}(Y_{j,1},\ldots,Y_{j,s_j}))$ is $\underline{t}$. Lemma 3 together with formula (6) and restriction (3) imply that for any $g \in G$ we have

$$g\cdot f = \sum_j \lambda_j \Phi^L(\mathrm{bpf}_{T_j}(Y_{j,1},\ldots,Y_{j,s_j}))\left(\prod_{v\in\mathcal{Q}_0,\, v<\boldsymbol{i}(v)} \det(g_v)^{w_{j,\boldsymbol{i}(v)}-w_{j,v}}\right).$$

It is not difficult to see that $w_{j,\boldsymbol{i}(v)} - w_{j,v}$ does not depend on j, therefore we denote it by μ_v. Hence,

$$g\cdot f = \prod_{v\in\mathcal{Q}_0,\, v<\boldsymbol{i}(v)} \det(g_v)^{\mu_v}\cdot f = f,$$

and the statement of the theorem follows immediately.

7 Reduction

Consider an arbitrary mixed quiver setting $\mathfrak{Q} = (\mathcal{Q},\boldsymbol{n},\boldsymbol{g},\boldsymbol{h},\boldsymbol{i})$ satisfying (3). As usual, define $G = G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$ and $H = H(\mathcal{Q},\boldsymbol{n},\boldsymbol{h})$.

Definition (of $\mathfrak{Q}^R$). Let a mixed quiver setting $\mathfrak{Q}^R = (\mathcal{Q}^R,\boldsymbol{n}^R,\boldsymbol{g}^R,\boldsymbol{h}^R,\boldsymbol{i}^R)$ and a mapping $\Phi^R : K[H(\mathcal{Q}^R,\boldsymbol{n}^R,\boldsymbol{h}^R)] \to K[H]$ be a result of the following procedure. (Here the letter R stands for the word *reduction*). Denote indeterminates of $K[H(\mathcal{Q}^R,\boldsymbol{n}^R,\boldsymbol{h}^R)]$ by $\overline{x}_{ij}^{\beta}$, where $\beta \in \mathcal{Q}_1^R$, and denote the corresponding generic matrix by $\overline{X}_\beta = (\overline{x}_{ij}^{\beta})$. Initially assume that $\mathfrak{Q}^R$ is $\mathfrak{Q}$ and for every $\alpha \in \mathcal{Q}_1$ we have $\Phi^R(\overline{X}_\alpha) = X_\alpha$. Afterwards we change $\mathfrak{Q}^R$ and Φ^R by performing steps a)–c).

a) For every $v \in \mathcal{Q}_0$ such that $\boldsymbol{g}_v$ is O or Sp we add a new vertex $\overline{v}$ to $\mathcal{Q}^R$ and set $\boldsymbol{i}^R(v) = \overline{v} > v$, $\boldsymbol{n}^R_{\overline{v}} = \boldsymbol{n}_v$, $\boldsymbol{g}^R_v = \boldsymbol{g}^R_{\overline{v}} = GL$. Also we add two new arrows β_v, γ_v to $\mathcal{Q}^R$ such that $\beta_v' = v$, $\beta_v'' = \overline{v}$, $\gamma_v' = \overline{v}$, $\gamma_v'' = v$. By definition, if $\boldsymbol{g}_v = O$, then $\Phi^R(\overline{X}_{\beta_v}) = \Phi^R(\overline{X}_{\gamma_v}) = E(\boldsymbol{n}_v)$, otherwise $\Phi^R(\overline{X}_{\beta_v}) = \Phi^R(\overline{X}_{\gamma_v}) = J(\boldsymbol{n}_v)$.

b) For every $v \in \mathcal{Q}_0$ with $\boldsymbol{g}_v = SO$ we add a new vertex $\overline{v}$ to $\mathcal{Q}^R$ and set $\boldsymbol{i}^R(v) = \overline{v} > v$, $\boldsymbol{n}^R_{\overline{v}} = \boldsymbol{n}_v$, $\boldsymbol{g}^R_v = \boldsymbol{g}^R_{\overline{v}} = SL$. Also we add a new arrow β_v to $\mathcal{Q}^R$ such that $\beta'_v = v$, $\beta''_v = \overline{v}$. By definition, $\Phi^R(\overline{X}_{\beta_v}) = E(\boldsymbol{n}_v)$;

c) Define $\boldsymbol{h}^R_\beta = M$ for all $\beta \in \mathcal{Q}^R_1$.

Note that $\mathcal{Q}_0 \subset \mathcal{Q}^R_0$, $\mathcal{Q}_1 \subset \mathcal{Q}^R_1$. For short, we write G^R for $G(\boldsymbol{n}^R, \boldsymbol{g}^R, \boldsymbol{i}^R)$ and H^R for $H(\mathcal{Q}^R, \boldsymbol{n}^R, \boldsymbol{h}^R)$.

Let $\psi : G \to G^R$ be the natural embedding, where $\psi(g)_v = g_v$ for $v \in \mathcal{Q}_0$, and $\psi(g)_{\overline{v}} = (g_v^{-1})^t$ for $v \in \mathcal{Q}_0$ with $\boldsymbol{g}_v \in \{O, Sp, SO\}$. It is not difficult to see that $g \cdot \Phi^R(\overline{X}_\beta) = \Phi^R(\psi(g) \cdot \overline{X}_\beta)$ for all $g \in G$ and $\beta \in \mathcal{Q}^R_1$. Hence

$$g \cdot \Phi^R(f) = \Phi^R(\psi(g) \cdot f) \tag{8}$$

for all $f \in K[H^R]$. Thus $\Phi^R(f)$ lies in $K[H]^G$ for all f from $K[H^R]^{G^R}$.

Theorem 7 *The restriction* $\Phi^R : K[H^R]^{G^R} \to K[H]^G$ *is a surjective mapping.*

Since a generating system for $K[H^R]^{G^R}$ is known (Theorem 2), Theorem 7 gives us a generating system for $K[H]^G$. In order to prove this theorem we need some facts from the theory of modules with good filtrations.

7.1 Good filtrations

Consider an affine algebraic group G with the coordinate ring $K[G]$, a closed subgroup B of G, and a rational B-module A. The tensor product $K[G] \otimes A$ over K is naturally a rational $G \times B$ module with respect to the action $(g, b) \cdot (f \otimes a) = f^{(g,b)} \otimes ba$ for $g \in G$, $b \in B$, $f \in K[G]$, $a \in A$, and $f^{(g,b)}(x) = f(g^{-1}xb)$. The set of B-fixed points of $K[G] \otimes A$ is a rational G-module, called the *induced* module $\mathrm{Ind}_B^G A = (K[G] \otimes A)^B$ (see, for example, [19]).

By a *good filtration* of a rational G-module M we mean an ascending chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ of submodules with $\cup_{i \in \mathbb{N}} M_i = M$ and for $i > 0$ the module M_{i+1}/M_i is either zero or is induced from a one-dimensional module over a fixed Borel subgroup of G.

- An affine G-variety H is called *good* if its coordinate ring $K[H]$ has a good filtration, where, as usual, G acts on $K[H]$ by the rule $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for $g \in G$, $f \in K[H]$, and $h \in H$.

- A pair of affine G-varieties (H_2, H_1) such that H_1 is a closed subvariety of H_2 given by an ideal I, i.e., $K[H_1] = K[H_2]/I$, is called a *good G-pair*, if H_2 is good, and I has a good filtration.

We list some standard properties of modules with good filtrations (see [9], [25]).

Theorem 8 *a) If*

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of G-modules and L has a good filtration, then

$$0 \to L^G \to M^G \to N^G \to 0$$

is exact.

b) If $M \subseteq N$ are modules with good filtrations, then N/M also has a good filtration.

c) If M, N are G-modules with good filtrations, then $M \otimes N$, considered under the diagonal action of G, also has a good filtration.

Corollary 3 *If (H_2, H_1) is a good G-pair, then H_1 is a good G-module and the mapping $K[H_2]^G \to K[H_1]^G$, induced by the natural surjection $K[H_2] \to K[H_1]$, is also a surjection.*

The following two lemmas will help us to construct good pairs.

Lemma 7 *a) Let M be a $GL(n)$-module. Then M has a good $GL(n)$-filtration if and only if M has a good $SL(n)$-filtration.*

b) Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ be an ascending chain of submodules such that M_{i+1}/M_i has a good filtration for any $1 \leq i < r$. Then M has a good filtration.

c) Let M be a $G_1 \times G_2$-module such that G_2 acts trivially on M. Then M has a good $G_1 \times G_2$-filtration if and only if M has a good G_1-filtration.

d) If (H_2, N) and (N, H_1) are good pairs, then (H_2, H_1) is a good pair.

e) If (H_1, N_1) and (H_2, N_2) are good G-pairs, then $(H_1 \oplus H_2, N_1 \oplus N_2)$ is a good G-pair, provided that G acts on the direct sums diagonally.

Proof. For parts b), c), d), respectively, see Proposition 1.2a (iv), Proposition 1.2e (ii), Lemma 1.3a (i) of [10], respectively. Since $SL(n)$ is the derived subgroup of $GL(n)$, part a) follows from part (i) of Lemma 1.4 of [12].

Now we prove part e). Part c) of Theorem 8 implies that $K[H_1 \oplus H_2] = K[H_1] \otimes K[H_2]$ has a good filtration. We have $K[N_i] = K[H_i]/I_i$ for $i = 1, 2$ and some ideals I_i. Consider the short exact sequence

$$0 \to I \to K[H_1] \otimes K[H_2] \to K[N_1] \otimes K[N_2] \to 0,$$

where $I = I_1 \otimes K[H_2] + K[H_1] \otimes I_2$. Since $I \,/\, I_1 \otimes I_2 = (I_1 \otimes K[N_2]) \oplus (K[N_1] \otimes I_2)$ has a good filtration and $I_1 \otimes I_2$ has a good filtration (see Theorem 8, c)), then I has a good filtration by part b). $\square$

Denote by $S_0^+(n)$ and $S_1^+(n)$, respectively, subsets of $S^+(n)$ consisting of invertible matrices and of matrices with the determinant 1, respectively. Similarly denote by $S_0^-(n)$ the subset of $S^-(n)$ that consists of invertible matrices. Let $\{1, *\}$ be the group of order two, where the symbol 1 stands for the identity element of the group. Given $\beta \in \{1, *\}$, a vector space $V = K^n$, and $g \in GL(n)$, we write

$$V^\beta = \left\{ \begin{array}{ll} V, & \beta = 1 \\ V^*, & \beta = * \end{array} \right. , \quad g^\beta = \left\{ \begin{array}{ll} g, & \beta = 1 \\ (g^{-1})^t, & \beta = * \end{array} \right. .$$

Lemma 8 *a) Let $GL(n)$ act on $K^{n\times n}$ by the formula $g \cdot A = gAg^t$ for $g \in GL(n)$, $A \in K^{n\times n}$. Then $(K^{n\times n}, S^+(n))$, $(K^{n\times n}, S^-(n))$, $(GL(n), S_0^+(n))$, and $(GL(n), S_0^-(n))$ are good $GL(n)$-pairs.*

b) Let the group $SL(n)$ act on $K^{n\times n}$ in the same manner as in part a). Then $(S^+(n), S_1^+(n))$ is a good $SL(n)$-pair.

c) Let $GL(n)$ act on $H = K^{n\times n} \oplus K^{n\times n}$ by the formula $g \cdot (A_1, A_2) = (gA_1g^t, (g^{-1})^t A_2 g^{-1})$ for $g \in GL(n)$ and $(A_1, A_2) \in H$. Then $(H, \{(A_1, A_2) \in H \,|\, A_1A_2 = E\})$ is a good $GL(n)$-pair.

d) Let $V = (K^n)^\beta$ and $W = (K^m)^\gamma$ for $\beta, \gamma \in \{1, *\}$. If $GL(n) \times GL(m)$ acts on $M = K^{m\times n} \simeq \mathrm{Hom}_K(V, W)$ in the natural way, i.e., $(g_1, g_2) \cdot A = g_2^\gamma A (g_1^\beta)^{-1}$, then M is a good $GL(n) \times GL(m)$-module.

Proof. For part a) see Lemma 1.3 of [33] and its proof. For part c) see the reasoning at the end of Section 1 of [33].

To prove part b) let $K[S^+(n)] = K[x_{ij}\,|\,1 \leq i,j \leq n]$, where $x_{ij} = x_{ji}$ and consider an $n \times n$ matrix $X = (x_{ij})$ and an ideal $I \lhd K[S^+(n)]$ generated by $\det(X) - 1$. Obviously $K[S_1^+(n)] = K[S^+(n)]/I$. A mapping $f \mapsto f \cdot (\det(X) - 1)$ is an isomorphism of $SL(n)$-modules $K[S^+(n)]$ and I. This fact together with part a) of this lemma, part a) of Lemma 7 and Corollary 3 imply that $S^+(n)$ is a good $SL(n)$-module and I has a good $SL(n)$-filtration.

To prove part d) notice that $K[M] \simeq S(W^* \otimes V)$ as $GL(n) \times GL(m)$-modules (for example, see Lemma 1 of [23]). Hence it is enough to show that $S^r(W^* \otimes V)$ has a good filtration for every $r > 0$. Consider ABW-filtration for $S^r(W^* \otimes V)$ (see Section 1.3 of [35] or Theorem 2.4 of [7]). Its factors are $L_\lambda(W^*) \otimes L_\lambda(V)$, where $L_\lambda(V)$ denotes a Schur module. The module $L_\lambda(V)$ has a good $GL(n)$-filtration and $L_\lambda(W^*)$ has a good $GL(m)$-filtration. By part c) of Lemma 7 both of these modules have good $GL(n) \times GL(m)$-filtrations. Part c) of Theorem 8 concludes the proof. $\square$

We have the following version of *Frobenius reciprocity*.

Lemma 9 *(c.f. Lemma 8.1 of [19]) If G is a closed subgroup of an algebraic group C and H is an affine C-variety (i.e. C acts rationally on H), then the algebra of invariants $(K[H] \otimes K[C/G])^C$ is isomorphic to $K[H]^G$, where C acts on C/G by left multiplication and it acts on the tensor product diagonally. The isomorphism is given by the mapping $a \otimes f \mapsto f(1_{C/G})a$ for $a \in K[H]$ and $f \in K[C/G]$.*

Frobenius reciprocity enables us to reduce investigation of the invariants of G to invariants of a bigger algebraic group C such that it is determined by smaller number of equations than G and it is more simple to calculate its invariants than invariants of G.

7.2 Proof of Theorem 7

We split the proof into several lemmas. Let us recall that $H = H(\mathcal{Q}, \boldsymbol{n}, \boldsymbol{h}) = \bigoplus_{\alpha \in \mathcal{Q}_1} H_\alpha$, where

a) $H_\alpha = K^{\boldsymbol{n}_{\alpha'} \times \boldsymbol{n}_{\alpha''}}$, if $\boldsymbol{h}_\alpha = M$;

b) $H_\alpha = S^+(\boldsymbol{n}_{\alpha'})$ and $\boldsymbol{g}_{\alpha'}$ is O or SO, if $\alpha' = \alpha''$ and $\boldsymbol{h}_\alpha = S^+$;

c) $H_\alpha = L^+(\boldsymbol{n}_{\alpha'})$ and $\boldsymbol{g}_{\alpha'} = Sp$, if $\alpha' = \alpha''$ and $\boldsymbol{h}_\alpha = L^+$;

d) $H_\alpha = S^+(\boldsymbol{n}_{\alpha'})$, $\boldsymbol{i}(\alpha') = \alpha''$, and $\boldsymbol{g}_{\alpha'}$ is GL or SL, if $\alpha' \neq \alpha''$ and $\boldsymbol{h}_\alpha = S^+$;

e) analogues statements are true if we substitute S^- for S^+ in parts b), d) and if we substitute L^- for L^+ in part c).

Denote

$$H_1 = \bigoplus_{\alpha\in\mathcal{Q}_1,\, \boldsymbol{h}_\alpha\neq L^+,L^-} H_\alpha \bigoplus_{\alpha\in\mathcal{Q}_1,\, \boldsymbol{h}_\alpha=L^+} S^+(\boldsymbol{n}_{\alpha'}) \bigoplus_{\alpha\in\mathcal{Q}_1,\, \boldsymbol{h}_\alpha=L^-} S^-(\boldsymbol{n}_{\alpha'}).$$

and define the action of G on H_1 by

$$(g\cdot h)_\alpha = \left\{ \begin{array}{ll} g_{\alpha'} h_\alpha g_{\alpha'}^t, & \text{if } \boldsymbol{h}_\alpha \neq M \text{ and } \alpha \text{ is a loop} \\ g_{\alpha'} h_\alpha g_{\alpha''}^{-1}, & \text{otherwise} \end{array} \right., \tag{9}$$

where $g=(g_v)_{v\in\mathcal{Q}_0}$ lies in G, $h=(h_\alpha)_{\alpha\in\mathcal{Q}_1}$ lies in H_1, and $\alpha\in\mathcal{Q}_1$. As a consequence of the following remark the spaces H and H_1 are isomorphic as G-modules.

Remark 1 *Let $g\in Sp(n)$, $A\in L^+(n)$, and $B\in S^+(n)$. Define the action of $Sp(n)$ on $L^+(n)$ by $g\cdot A = gAg^{-1}$ and its action on $S^+(n)$ by $g\cdot B = gBg^t$. Then the mapping $L^+(n)\to S^+(n)$ given by $A\mapsto AJ$ is an isomorphism of $Sp(n)$-modules. The assertion remains valid if we replace L^+ and S^+, respectively, by L^- and S^-, respectively.*

Rewrite G as $G = P\times\prod G_v$, where v ranges over vertices of $\mathcal{Q}$ such that $\boldsymbol{g}_v\in\{O, Sp, SO\}$. The group G is a subgroup of

$$C = P\times \prod_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=O,Sp} GL(n_v) \prod_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=SO} SL(n_v).$$

Notice that H_1 is also a C-module, where the action is defined by (9). Frobenius reciprocity (see Lemma 9) gives the isomorphism

$$(K[H_1]\otimes K[C/G])^C \simeq K[H_1]^G.$$

Consider the C-module

$$W_1 = \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=O} S_0^+(\boldsymbol{n}_v) \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=Sp} S_0^-(\boldsymbol{n}_v) \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=SO} S_1^+(\boldsymbol{n}_v),$$

where $S_0^+(n)$, $S_0^-(n)$, and $S_1^+(n)$ were defined just before Lemma 8. Here the action is given by

$$(g\cdot w)_v = g_v w_v g_v^t \text{ for all } v\in\mathcal{Q}_0 \text{ with } \boldsymbol{g}_v\in\{O, Sp, SO\}, \tag{10}$$

where $g \in C$ and $w = (w_v)_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v\in\{O,Sp,SO\}}$ belongs to W_1. The mapping $\pi_1 : C/G \to W_1$ given by

$$\pi_1(\overline{g}) = \left\{ \begin{array}{ll} g_v g_v^t, & \text{if } g_v \text{ is } O \text{ or } SO \\ g_v J g_v^t, & \text{if } g_v = Sp \end{array} \right.$$

($\overline{g}$ is the equivalence class of $g \in C$, $v \in \mathcal{Q}_0$, and $\boldsymbol{g}_v \in \{O, Sp, SO\}$) is well defined.

Lemma 10 *The mapping π_1 is a C-equivariant isomorphism of the algebraic groups C/G and W_1.*

Proof. Mappings

$$GL(n)/O(n) \to S_0^+(n),\ \overline{g} \mapsto gg^t \text{ and}$$

$$GL(n)/Sp(n) \to S_0^-(n),\ \overline{g} \mapsto gJg^t$$

are $GL(n)$-equivariant isomorphisms of the algebraic groups, where $GL(n)$ acts on the left hand sides by left multiplication and it acts on the right hand side by the formula $g \cdot A = gAg^t$ (see Lemma 1.2 of [33]). Similarly, a mapping

$$SL(n)/SO(n) \to S_1^+(n),\ \overline{g} \mapsto gg^t,$$

is an $SL(n)$-equivariant isomorphisms of the algebraic groups, where the actions are the same as above. This completes the proof. $\square$

The space H_1 is contained in

$$H_2 = \bigoplus_{\alpha\in\mathcal{Q}_1} K^{\boldsymbol{n}_{\alpha'}\times\boldsymbol{n}_{\alpha''}}$$

and the space W_1 is contained in

$$W_2 = \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v \text{ is } O \text{ or } Sp} GL(\boldsymbol{n}_v) \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v = SO} S^+(\boldsymbol{n}_v).$$

Define the action of C on H_2 and W_2, respectively, by the formulas (9) and (10), respectively.

Lemmas 7, 8 imply the following result.

Lemma 11 *The C-pair $(H_2 \oplus W_2, H_1 \oplus W_1)$ is a good one.*

We refer to the elements of the space

$$W_3 = \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=O,Sp} (K^{\boldsymbol{n}_v\times\boldsymbol{n}_v}\oplus K^{\boldsymbol{n}_v\times\boldsymbol{n}_v}) \bigoplus_{v\in\mathcal{Q}_0,\, \boldsymbol{g}_v=SO} K^{\boldsymbol{n}_v\times\boldsymbol{n}_v}$$

as $w = (w_v, w_{\overline{u}})_{v,u\in\mathcal{Q}_0,\, \boldsymbol{g}_v\in\{O,Sp,SO\},\, \boldsymbol{g}_u\in\{O,Sp\}}$. Endow W_3 with the structure of C-module by

$$(g\cdot w)_v = g_v w_v g_v^t \text{ for all } v\in\mathcal{Q}_0 \text{ with } \boldsymbol{g}_v\in\{O,Sp,SO\},$$

$$(g\cdot w)_{\overline{u}} = (g_u^{-1})^t w_{\overline{u}} g_u^{-1} \text{ for all } u\in\mathcal{Q}_0 \text{ with } \boldsymbol{g}_u\in\{O,Sp\}$$

for $g\in C$. Define the embedding

$$\pi_2 : W_2 \to W_3$$

of C-spaces by $\pi_2(w)_v = w_v$ and $\pi_2(w)_{\overline{u}} = w_u^{-1}$.

The following lemma is a consequence of Lemmas 7, 8.

Lemma 12 *The C-pair $(H_2\oplus W_3, H_2\oplus\pi_2(W_2))$ is a good one.*

Using Corollary 3 together with Lemmas 10, 11, 12, and Frobenius reciprocity (see Lemma 9) we obtain a surjection

$$\Phi : K[H_2\oplus W_3]^C \to K[H]^G.$$

Consider the mixed quiver setting $\mathfrak{Q}^R$. Remove each loop $\alpha\in\mathcal{Q}_1^R$ with $\boldsymbol{h}_\alpha\neq M$ and replace it by a new arrow β_α with $\beta'_\alpha = v$, $\beta''_\alpha = \overline{v}$, where $\alpha'=\alpha''=v$. Denote the resulting quiver by $\mathcal{Q}^0$ and set $\mathfrak{Q}^0 = (\mathcal{Q}^0, \boldsymbol{n}^R, \boldsymbol{g}^R, \boldsymbol{h}^0, \boldsymbol{i}^R)$, where $\boldsymbol{h}^0_\beta = M$ for all $\beta\in\mathcal{Q}_1^0$. For short, we write G^0 for $G(\boldsymbol{n}^R, \boldsymbol{g}^R, \boldsymbol{i}^R)$ and write H^0 for $H(\mathcal{Q}^0, \boldsymbol{n}^R, \boldsymbol{h}^0)$. Denote indeterminates of $K[H^0]$ by $\overline{x}_{ij}^\beta$, where $\beta\in\mathcal{Q}_1^0$, and denote the corresponding generic matrix by $\overline{X}_\beta = (\overline{x}_{ij}^\beta)$. Then $C = G^0$ and G^0-modules $H_2\oplus W_3$ and H^0 are equal.

Define a mapping $\Phi^0 : K[H^0]\to K[H]$ by

$$\Phi^0(\overline{X}_\gamma) = \left\{\begin{array}{cl} X_\alpha, & \text{if } \gamma=\beta_\alpha \text{ for a loop } \alpha\in\mathcal{Q}_1 \text{ with } \boldsymbol{h}_\alpha\in\{S^+,S^-\} \\ X_\alpha J(\boldsymbol{n}_v), & \text{if } \gamma=\beta_\alpha \text{ for a loop } \alpha\in\mathcal{Q}_1 \text{ with } \boldsymbol{h}_\alpha\in\{L^+,L^-\} \\ \Phi^R(\overline{X}_\beta), & \text{if } \gamma\in\mathcal{Q}_1^R\cap\mathcal{Q}_1^0 \end{array}\right.$$

for $\gamma\in\mathcal{Q}_1^0$. It is not difficult to see that the restriction of Φ^0 to G^0-invariants coincides with Φ.

Let $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$, where $(T,(Y_1,\ldots,Y_s))$ is a $\mathfrak{Q}^0$-tableau with substitution of a weight $\underline{w}$, be a G^0-invariant. Then there is a path $\mathfrak{Q}^R$-tableau with substitution $(D,(Z_1,\ldots,Z_s))$ such that

$$\Phi^0(\mathrm{bpf}_T(Y_1,\ldots,Y_s)) = \Phi^R(\mathrm{bpf}_D(Z_1,\ldots,Z_s))$$

and the weight of $(D,(Z_1,\ldots,Z_s))$ is $\underline{w}$. (Note that $\mathcal{Q}_0^0=\mathcal{Q}_0^R$.) Lemma 3 implies that $\mathrm{bpf}_D(Z_1,\ldots,Z_s)$ is a G^R-invariant. Theorem 2 together with (8) shows that

$$\Phi^0(K[H^0]^{G^0}) \subset \Phi^R(K[H^R]^{G^R}) \subset K[H]^G.$$

The fact that Φ is surjective completes the proof of Theorem 7.

8 Path $\mathfrak{Q}$-tableaux with substitutions and good $\mathfrak{Q}$-tableaux with substitutions

Consider a tableau with substitution $(T,(X_1,\ldots,X_s))$ of dimension $\underline{n}\in\mathbb{N}^m$ and numbers $1\leq q_1<q_2\leq m$ with $n_{q_1}=n_{q_2}$. Let us recall some concepts introduced in [24].

Denote by $\mathcal{M}$ the monoid freely generated by letters $x_1,x_2,\ldots,\ x_1^t,x_2^t,\ldots$ Let $\mathcal{M}_T$ be the submonoid of $\mathcal{M}$, generated by $x_1,\ldots,x_s,\ x_1^t,\ldots,x_s^t$. For short, we will write $1,\ldots,s,1^t,\ldots,s^t$ instead of $x_1,\ldots,x_s,x_1^t,\ldots,x_s^t$. Given $a\in T$, we consider $\varphi(a)\in\{1,\ldots,s\}$ as an element of $\mathcal{M}_T$.

Given $u\in\mathcal{M}_T$, define the matrix X_u by the following rules:

- $X_{j^t}=X_j^t$ for any $1\leq j\leq s$;
- $X_{vw}=\begin{cases} X_vX_w, & \text{if the product of these matrices is well defined}\\ 0, & \text{otherwise}\end{cases}$
  for $v,w\in\mathcal{M}_T$.

For an arrow $a\in T$ denote by a^t the *transpose arrow*, i.e., by definition $(a^t)''=a'$, $(a^t)'=a''$, $''(a^t)={}'a$, $'(a^t)={}''a$, $\varphi(a^t)=\varphi(a)^t\in\mathcal{M}_T$. Obviously, $(a^t)^t=a$.

We write $a\overset{t}{\in}T$ if $a\in T$ or $a^t\in T$.

Definition (of paths in T). We say that $a_1,a_2\overset{t}{\in}T$ are *successive* in T (with respect to q_1 and q_2), if $a_1',a_2''\in\{q_1,q_2\}$, $a_1'\neq a_2''$, ${}'a_1={}''a_2$.

A word $a=a_1\cdots a_r$ with $a_1,\ldots,a_r\overset{t}{\in}T$, is called a *path* in T with respect to columns q_1 and q_2, if $a_i,\ a_{i+1}$ are successive (with respect to q_1 and q_2) for

any $1 \le i \le r-1$. In this case by definition $\varphi(a) = \varphi(a_1)\cdots\varphi(a_r) \in \mathcal{M}_T$ and $a^t = a_r^t \cdots a_1^t$ is a path in T; we denote $a_r', 'a_r, a_1'', ''a_1$, respectively, by $a', 'a, a'', ''a$, respectively. If the columns q_1 and q_2 are fixed, we refer to a as a path in T. A path $a_1\cdots a_r$ is *closed* if a_r, a_1 are successive; in particular, $a_1'', a_r' \in \{q_1, q_2\}$.

Given $\xi \in \mathcal{S}_{n_{q_1}}$ we will also use the notations $(T^{\xi}, (Y_1,\ldots,Y_s))$ and $(\widetilde{T}^{\xi}, Y_{\widetilde{T}^{\xi}})$, respectively, for the tableaux with substitutions of dimensions $\underline{n}$ and $\underline{d}$, respectively, defined in Section 4 of [24]. Here $Y_{\widetilde{T}^{\xi}} = (Z_1,\ldots,Z_h)$ for some matrices $Z_1,\ldots,Z_h$.

Throughout this section we assume that $\mathfrak{Q} = (\mathcal{Q}, \boldsymbol{n}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i})$ is a mixed quiver setting satisfying (3) and

$$\boldsymbol{g}_v \neq \mathit{Sp} \text{ for all } v \in \mathcal{Q}_0. \tag{11}$$

Consider an arrow $\beta \in \mathcal{Q}_1^D$. If $\beta \in \mathcal{Q}_1$ and $\boldsymbol{h}_\beta = M$, then $\beta^t \in \mathcal{Q}_1^D$ is given by the definition of a mixed double quiver setting. Otherwise, we set

$$\beta^t = \left\{ \begin{array}{ll} \alpha, & \text{if } \beta = \alpha^t,\ \boldsymbol{h}_\alpha = M, \text{ and } \alpha \in \mathcal{Q}_1 \\ \beta, & \text{if } \beta \in \mathcal{Q}_1 \text{ and } \boldsymbol{h}_\beta \neq M \end{array} \right. .$$

If $\beta = \beta_1\cdots\beta_r$ is a path in $\mathcal{Q}^D$, denote by $\beta^t = \beta_r^t\cdots\beta_1^t$ a path in $\mathcal{Q}^D$. Condition (11) implies that $\Phi^D(X_{\beta_1^t}\cdots X_{\beta_r^t}) = \Phi^D(X_{\beta_r}\cdots X_{\beta_1})^t$.

Definition (of good $\mathfrak{Q}$-tableaux with substitutions). A tableau with substitution $(T, (Y_1,\ldots,Y_s))$ of dimension $\underline{n} \in \mathbb{N}^m$ is called a *good $\mathfrak{Q}$-tableau with substitution*, if for some *weight* $\underline{w} = (w_1,\ldots,w_l) \in \mathbb{N}^l$ and the distribution W, determined by $\underline{w}$, one has

- $\underline{n} = (\underbrace{\boldsymbol{n}_1,\ldots,\boldsymbol{n}_1}_{w_1},\ldots,\underbrace{\boldsymbol{n}_l\ldots,\boldsymbol{n}_l}_{w_l})$;
- if $a \in T$, then there exists a path $\beta = \beta_1\cdots\beta_r$ in $\mathcal{Q}^D$ (where $\beta_1,\ldots,\beta_r \in \mathcal{Q}_1^D$) such that $Y_{\varphi(a)} = \Phi^D(X_{\beta_r}\cdots X_{\beta_1})$, $W|a'| = \boldsymbol{i}(\beta'')$, and $W|a''| = \beta'$.

Remark 2 *a) Any path $\mathfrak{Q}$-tableau with substitution is also a good $\mathfrak{Q}$-tableau with substitution.*

b) For every good $\mathfrak{Q}$-tableau with substitution $(T, (Y_1,\ldots,Y_s))$ of a weight $\underline{w}$ there is a path $\mathfrak{Q}^D$-tableau with substitution $(T, (Z_1,\ldots,Z_s))$ of weight $\underline{w}$ such that for every $a \in T$ we have the equality $Z_{\varphi(a)} = X_{\beta_r}\cdots X_{\beta_1}$ for a path $\beta_1\cdots\beta_r$ from the definition. Moreover, $\mathrm{bpf}_T(Y_1,\ldots,Y_s) = \Phi^D(\mathrm{bpf}_T(Z_1,\ldots,Z_s))$.

Lemma 13 *Consider a good $\mathfrak{Q}$-tableau with substitution $(T,(Y_1,\ldots,Y_s))$ of a weight $\underline{w}$, that determines the distribution W, and let $\underline{n}\in\mathbb{N}^m$ be the dimension of $(T,(Y_1,\ldots,Y_s))$. Assume that numbers $1\leq q_1<q_2\leq m$ satisfy $\boldsymbol{i}(W|q_1|)=W|q_2|$ and $b=b_1\cdots b_p$ (where $b_1,\ldots,b_p\overset{t}{\in}T$) is a path in T with respect to columns q_1 and q_2.*

Then there is a path $\beta=\beta_1\cdots\beta_r$ in $\mathcal{Q}^D$ (where $\beta_1,\ldots,\beta_r\in\mathcal{Q}_1^D$) such that $Y_{\varphi(b)}=\Phi^D(X_{\beta_r}\cdots X_{\beta_1})$, $W|b'|=\boldsymbol{i}(\beta'')$, and $W|b''|=\beta'$. In particular, if b is a closed path in T, then β is a closed path in $\mathcal{Q}^D$.

Proof. Obviously, it is enough to prove the lemma for $p=2$. For $j\in\{1,2\}$, the condition $b_j\overset{t}{\in}T$ implies that there is an $a_j\in T$ with $b_j\in\{a_j,a_j^t\}$. By the definition of a good $\mathfrak{Q}$-tableau with substitution, there is a path $\alpha_j=\alpha_{j,1}\cdots\alpha_{j,r_j}$ in $\mathcal{Q}^D$ (where $\alpha_{j,1},\ldots,\alpha_{j,r_j}\in\mathcal{Q}_1^D$) such that $Y_{\varphi(a_j)}=\Phi^D(X_{\alpha_{j,r_j}}\cdots X_{\alpha_{j,1}})$, $W|a_j'|=\boldsymbol{i}(\alpha_j'')$, and $W|a_j''|=\alpha_j'$.

Assume $b_1=a_1$ and $b_2=a_2^t$. Then $\{q_1,q_2\}=\{a_1',a_2'\}$. Define $\beta=\alpha_2^t\alpha_1=\alpha_{2,r_2}^t\cdots\alpha_{2,1}^t\alpha_{1,1}\cdots\alpha_{1,r_1}=\beta_1\cdots\beta_r$, where $\beta_1,\ldots,\beta_r\in\mathcal{Q}_1^D$, $r=r_1+r_2$. Since $(\alpha_2^t)'=\boldsymbol{i}(\alpha_2'')=W|a_2'|=\boldsymbol{i}(W|a_1'|)=\alpha_1''$, we infer that β is a path in $\mathcal{Q}^D$. Obviously,

$$Y_{\varphi(b)}=Y_{\varphi(a_1)}Y_{\varphi(a_2)}^t=\Phi^D(X_{\alpha_{1,r_1}}\cdots X_{\alpha_{1,1}}X_{\alpha_{2,1}}^t\cdots X_{\alpha_{2,r_2}}^t)=\Phi^D(X_{\beta_r}\cdots X_{\beta_1}).$$

By the definitions of α_1, α_2, we have $W|b'|=W|b_2'|=W|a_2''|=\alpha_2'=\boldsymbol{i}(\beta'')$ and $W|b''|=W|b_1''|=W|a_1''|=\alpha_1'=\beta'$. This proves the claim for the given case.

The remaining cases can be treated analogously. □

Theorem 9 *Let $\mathfrak{Q}=(\mathcal{Q},\boldsymbol{n},\boldsymbol{g},\boldsymbol{h},\boldsymbol{i})$ be a mixed quiver setting satisfying conditions (3) and (11). Let $(T,(Y_1,\ldots,Y_s))$ be a path $\mathfrak{Q}$-tableau with substitution of a weight $\underline{w}$, that determines the distribution W. Then $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$ is a polynomial over K in $\Phi^D(\sigma_k(X_{\beta_r}\cdots X_{\beta_1}))$ and $\Phi^D(\mathrm{bpf}_D(Z_1,\ldots,Z_h))$, where*

1. *$\beta_1\cdots\beta_r$ ranges over closed paths in $\mathcal{Q}^D$ and $1\leq k\leq \boldsymbol{n}_{\beta_1''}$;*
2. *$(D,(Z_1,\ldots,Z_h))$ ranges over path $\mathfrak{Q}^D$-tableaux with substitutions of a weight $\underline{u}$ such that*
   - *a) for every $v\in\mathcal{Q}_0$ we have $u_v\leq w_v$ and $w_{\boldsymbol{i}(v)}-w_v=u_{\boldsymbol{i}(v)}-u_v$;*
   - *b) if $\boldsymbol{g}_v$ is GL or SL for $v\in\mathcal{Q}_0$, then $u_{\boldsymbol{i}(v)}=0$ or $u_v=0$;*
   - *c) if $\boldsymbol{g}_v$ is O or SO for $v\in\mathcal{Q}_0$, then $u_v\leq 1$ and $\boldsymbol{i}(v)=v$.*

Proof. We have that $(T,(Y_1,\ldots,Y_s))$ is a good $\mathfrak{Q}$-tableau with substitution of weight $\underline{w}$. If conditions b) and c) are valid for $\underline{u}=\underline{w}$, then the statement is trivial. Otherwise, there is a $v\in\mathcal{Q}_0$ such that $\boldsymbol{i}(v)\neq i$, $w_{\boldsymbol{i}(v)}\neq 0$, $w_v\neq 0$ or $\boldsymbol{i}(v)=v$, $w_v\geq 2$. Consider $1\leq q_1<q_2\leq m$ such that $W|q_1|=v$, $W|q_2|=\boldsymbol{i}(v)$. In particular, $\boldsymbol{i}(W|q_1|)=W|q_2|$ and $n_{q_1}=\boldsymbol{n}_v=\boldsymbol{n}_{\boldsymbol{i}(v)}=n_{q_2}$. Apply the decomposition formula (see Theorem 2 from [24]) to the q_1-th and the q_2-th columns of T. Then $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$ is a polynomial in $\sigma_k(Y_{\varphi(c)})$, $\mathrm{bpf}_D(Z_1,\ldots,Z_h)$, where

- c is a closed path in T^{ξ} for some $\xi\in\mathcal{S}_{n_{q_1}}$;
- $(D,(Z_1,\ldots,Z_h))=(\widetilde{T}^{\pi},Y_{\widetilde{T}^{\pi}})$ is a tableau with substitution of dimension $\underline{d}$ for some $\pi\in\mathcal{S}_{n_{q_1}}$.

Here $\underline{d}\in\mathbb{N}^{m-2}$ is obtained from $\underline{n}$ by eliminating the q_1-th and q_2-th coordinates. Applying Lemma 13 to paths of $\widetilde{T}^{\pi}$, we obtain that $(\widetilde{T}^{\pi},Y_{\widetilde{T}^{\pi}})$ is a good $\mathfrak{Q}$-tableau with substitution. Lemma 13 also imply that $\sigma_k(Y_{\varphi(c)})=\sigma_k(\Phi^D(X_{\beta_r}\cdots X_{\beta_1}))=\Phi^D(\sigma_k(X_{\beta_r}\cdots X_{\beta_1})$ for a closed path $\beta_1\cdots\beta_r$ in $\mathcal{Q}^D$.

Repeat this procedure for $(D,(Z_1,\ldots,Z_h))$ and so on. Finally we see that $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$ is a polynomial in $\Phi^D(\sigma_k(X_{\beta_r}\cdots X_{\beta_1}))$ and $\mathrm{bpf}_D(Z_1,\ldots,Z_h)$ such that condition 1 is valid and $(D,(Z_1,\ldots,Z_h))$ is a good $\mathfrak{Q}$-tableau with substitution of a weight $\underline{u}$ satisfying a), b), and c). Part b) of Remark 2 completes the proof. $\square$

9 Proof of Theorem 1

Consider a mixed quiver setting $\mathfrak{Q}=(\mathcal{Q},\boldsymbol{n},\boldsymbol{g},\boldsymbol{h},\boldsymbol{i})$ satisfying (3). As usual, define $G=G(\boldsymbol{n},\boldsymbol{g},\boldsymbol{i})$, $H=H(\mathcal{Q},\boldsymbol{n},\boldsymbol{h})$. In Lemma 4 we have shown that the elements from Theorem 1 are invariants.

Successive applications of Theorem 7 and Theorem 2 yield the generating system that consists of images of $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$ for some tableaux with substitutions $(T,(Y_1,\ldots,Y_s))$. Application of Theorem 9 to $\mathrm{bpf}_T(Y_1,\ldots,Y_s)$ gives the mixed quiver setting, which we denote by $\mathfrak{Q}^0=(\mathcal{Q}^0,\boldsymbol{n}^0,\boldsymbol{g}^0,\boldsymbol{h}^0,\boldsymbol{i}^0)$, and the mapping $\Phi^0: K[H^0]\to K[H]$, where H^0 stands for $H(\mathcal{Q}^0,\boldsymbol{n}^0,\boldsymbol{h}^0)$, such that

$$\mathcal{Q}^0_0=\mathcal{Q}^R_0=\mathcal{Q}_0\coprod\{\overline{v}\,|\,v\in\mathcal{Q}_0,\ \boldsymbol{g}_v\in\{O,Sp,SO\}\},$$

$$\mathcal{Q}^0_1=\mathcal{Q}_1\coprod\{\alpha^t\,|\,\alpha\in Q_0\}\coprod\{\alpha_v,\alpha_v^t\,|\,v\in\mathcal{Q}_0,v\leq\boldsymbol{i}(v)\}$$

$$\coprod\{\beta_v,\beta_v^t\,|\,v\in\mathcal{Q}_0,\boldsymbol{g}_v\in\{O,Sp,SO\}\}\coprod\{\gamma_v,\gamma_v^t\,|\,v\in\mathcal{Q}_0,\boldsymbol{g}_v\in\{O,Sp\}\},$$

$$\boldsymbol{i}^0(v) = \left\{ \begin{array}{cl} \boldsymbol{i}(v), & \text{if } \boldsymbol{g}_v \in \{GL, SL\} \\ \overline{v}, & \text{if } \boldsymbol{g}_v \in \{O, Sp, SO\} \end{array} \right. , \qquad \text{where } v \in \mathcal{Q}_0,$$

$$\boldsymbol{g}^0_v = \boldsymbol{g}^0_{\boldsymbol{i}^0(v)} = \left\{ \begin{array}{ll} GL, & \text{if } \boldsymbol{g}_v \in \{GL, O, Sp\} \\ SL, & \text{if } \boldsymbol{g}_v \in \{SL, SO\} \end{array} \right. , \qquad \text{where } v \in \mathcal{Q}_0,$$

$$\boldsymbol{h}^0_\beta = M \text{ for all } \beta \in \mathcal{Q}^0_1, \text{ and } \boldsymbol{n}^0 = \boldsymbol{n}^R.$$

Here β_v, β^t_v go from $\boldsymbol{i}^0(v) = \overline{v}$ to v, arrows γ_v, γ^t_v go in the opposite direction (i.e. from v to $\overline{v}$), and α_v, α^t_v are loops in $v, \boldsymbol{i}^0(v)$, respectively. Denote indeterminates of $K[H^0]$ by $\overline{x}^\beta_{ij}$ for $\beta \in \mathcal{Q}^0_1$ and denote the corresponding generic matrix by $\overline{X}_\beta = (\overline{x}^\beta_{ij})$. The mapping Φ^0 is determined by

$$\Phi^0(\overline{X}_{\alpha_v}) = E,$$

$$\Phi^0(\overline{X}_{\beta_v}) = \Phi^0(\overline{X}_{\gamma_v}) = \left\{ \begin{array}{ll} E, & \text{if } \boldsymbol{g}_v \text{ is } O \text{ or } SO \\ J, & \text{if } \boldsymbol{g}_v = Sp \end{array} \right. ,$$

$$\Phi^0(\overline{X}_\alpha) = X_\alpha \text{ for } \alpha \in Q_1,$$

$$\Phi^0(\overline{X}_{\beta^t}) = X^t_\beta \text{ for } \beta \in Q^0_1.$$

The algebra of invariants $K[H]^G$ is generated by the elements $\Phi^0(\mathrm{bpf}_T(Y_1, \ldots, Y_s))$ and $\Phi^0(\sigma_k(\overline{X}_{\gamma_r} \cdots \overline{X}_{\gamma_1}))$, where

- $(T, (Y_1, \ldots, Y_s))$ is a path $\mathfrak{Q}^0$-tableau with substitution of a weight $\underline{u}$, satisfying the conditions
  - a) if $\boldsymbol{g}^0_v = GL$ ($v \in \mathcal{Q}^0_0$), then $u_{\boldsymbol{i}^0(v)} = u_v = 0$,
  - b) if $\boldsymbol{g}^0_v = SL$ ($v \in \mathcal{Q}^0_0$), then $u_{\boldsymbol{i}^0(v)} = 0$ or $u_v = 0$.
- $\gamma_1 \cdots \gamma_r$ is a closed path in $\mathcal{Q}^0$, $1 \leq k \leq \boldsymbol{n}^0_{\gamma''_1}$.

Lemma 14 *Let $\gamma = \gamma_1 \cdots \gamma_r$ be a path in $\mathcal{Q}^0$ and $X = \Phi^0(\overline{X}_{\gamma_r} \cdots \overline{X}_{\gamma_1})$.*

a) If X is not a matrix over K, then there is a path $\beta = \beta_1 \cdots \beta_p$ in $\mathcal{Q}^D$ such that $X = \pm\Phi^D(X_{\beta_p} \cdots X_{\beta_1})$ and

$$\beta' = \left\{ \begin{array}{ll} \gamma', & \text{if } \gamma' \in \mathcal{Q}_0 \\ \boldsymbol{i}^0(\gamma'), & \text{if } \gamma' \notin \mathcal{Q}_0 \end{array} \right. , \qquad \beta'' = \left\{ \begin{array}{ll} \gamma'', & \text{if } \gamma'' \in \mathcal{Q}_0 \\ \boldsymbol{i}^0(\gamma''), & \text{if } \gamma'' \notin \mathcal{Q}_0 \end{array} \right. .$$

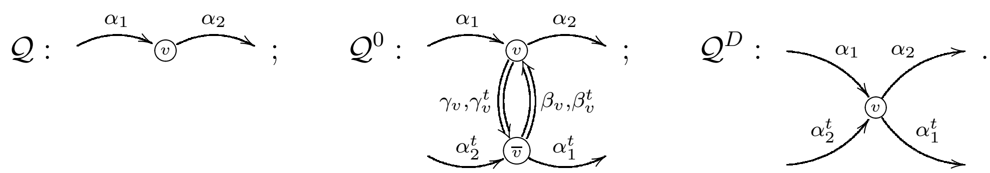
In particular, if γ is closed, then β is closed.

b) If X is a matrix over K, then there is a $v \in \mathcal{Q}_0$ such that $\gamma_i', \gamma_i'' \in \{v, \boldsymbol{i}^0(v)\}$ for any $1 \le i \le r$. Moreover,

if $\boldsymbol{g}_v$ is GL or SL, then for any $1 \le i \le r$ an arrow γ_i is a loop;

if $\boldsymbol{g}_v \neq Sp$, then $X = \pm E$, otherwise, X is $\pm E$ or $\pm J$.

Proof. **a)** Eliminate arrows α_v, α_v^t from the path γ and obtain a path in $\mathcal{Q}^0$. Then eliminate arrows $\beta_v, \beta_v^t, \gamma_v, \gamma_v^t$ to get the required path. To prove it, see the following pictures, where we depicted some vertex $v \in \mathcal{Q}_0$ with $\boldsymbol{g}_v \in \{O, Sp, SO\}$ and two arrows $\alpha_1, \alpha_2 \in \mathcal{Q}_1$.



b) This case is trivial. $\square$

The last lemma shows that we can reduce $\mathfrak{Q}^0$ to a mixed quiver setting $\mathfrak{Q}^1 = (\mathcal{Q}^1, \boldsymbol{n}^D, \boldsymbol{g}^D, \boldsymbol{h}^1, \boldsymbol{i}^D)$, which almost coincides with $\mathfrak{Q}^D$. To do so, define $\mathcal{Q}^1_0 = \mathcal{Q}^D_0 = \mathcal{Q}_0$ and $\mathcal{Q}^1_1 = \mathcal{Q}^D_1 \coprod \{\delta_1, \ldots, \delta_q\}$, where δ_i is a loop and $q \ge 0$. Then consider a mapping $\Phi^1 : K[H(\mathcal{Q}^0, \boldsymbol{n}^0, \boldsymbol{h}^0)] \to K[H]$ such that

$$\text{if } \beta \in \mathcal{Q}_0, \text{ then } \Phi^1(X_\beta) = \Phi^D(X_\beta), \text{ otherwise } \Phi^1(X_\beta) \text{ is } \pm E \text{ or } \pm J,$$

and assume that for every path $\mathfrak{Q}^0$-tableau with substitution $(T, (Y_1, \ldots, Y_s))$ satisfying the property formulated before Lemma 14 there is a path $\mathfrak{Q}^1$-tableau with substitution $(T, (Z_1, \ldots, Z_s))$ satisfying

a) $\Phi^0(\mathrm{bpf}_T(Y_1, \ldots, Y_s)) = \pm\Phi^1(\mathrm{bpf}_T(Z_1, \ldots, Z_s))$;

b) if the weight of $(T, (Y_1, \ldots, Y_s))$ is $\underline{u}$, then the weight of $(T, (Z_1, \ldots, Z_s))$ is $\underline{w}$, where for every $v \in \mathcal{Q}_0$ we have

$$w_v = \left\{ \begin{array}{cl} u_v, & \text{if } \boldsymbol{g}_v \text{ is } GL \text{ or } SL \\ u_v + u_{\overline{v}}, & \text{if } \boldsymbol{g}_v \in \{O, Sp, SO\} \end{array} \right. .$$

Consider a vertex $v \in \mathcal{Q}_0$. There are three possibilities.

- If $\boldsymbol{g}_v \in \{GL, O, Sp\}$, then equalities $w_{\boldsymbol{i}(v)} = w_v = 0$ imply that δ_i is not a loop in v for all i.

- If $\boldsymbol{g}_v$ is SL, then, since $w_v = 0$ or $w_{\boldsymbol{i}(v)} = 0$, we see that δ_i is not a loop in v for all i.
- Let $\boldsymbol{g}_v$ be SO and let δ_i be a loop in v for some i. Then $\Phi^1(X_{\delta_i}) = \pm E$. Theorem 9 together with the facts

  a) $\mathrm{bpf}_T(Y_1, \ldots, Y_s) = 0$ for any tableau with substitution $(T, (Y_1, \ldots, Y_s))$ with $Y_{\varphi(a)} = E$ and $a' = a''$ for some $a \in T$;

  b) every $\mathfrak{Q}^{DD}$-tableau with substitution, where $\mathfrak{Q}^{DD}$ is a mixed double quiver setting of $\mathfrak{Q}^D$, is also a path $\mathfrak{Q}^D$-tableau with substitution;

  completes the proof.

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